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GMM and IV Estimation of Dynamic Panel Models with Heterogeneous Trend*

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Abstract

In this paper, we consider the generalized method of moment (GMM) and simple instrumental variable (IV) type estimation of dynamic panel data models with both individual-specific effects and heterogeneous time trend. We consider the forward demeaning (FOD) proposed by Hayakawa et al (2017) and the double first difference (FD) to remove both the individual-specific effects and heterogeneous trend. We establish the asymptotic properties of the GMM estimation of the lag coefficient and find that the GMM estimation using FOD is asymptotically biased of order $\sqrt{\frac{T}{N}}$, while the GMM using FD is asymptotically biased of order $\sqrt{\frac{T^3}{N}}$. We also establish the asymptotic unbiasedness of the simple IV estimation. Monte Carlo simulations confirm our findings in this paper.

Keywords: Dynamic panel data model, Individual-specific effects, Time trend, Generalized method of moments, Instrumental Variable, Forward demeaning, First difference
JEL classification: C01, C13, C23

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1 Introduction

In this paper, we consider the dynamic panel data models with both individual-specific effects and heterogeneous time trend. One of the unique features of the dynamic panel models is that the presence of unobserved individual-specific effects creates the correlation between all past, current and future observations (Hsiao (2014)). However, the individual-specific effects appears linearly in the model. In principle, any differencing method that preserves the linear structure of the model can eliminate the time-invariant individual-specific effects. After elimination of individual-specific effects, another unique feature of panel dynamic models is that lagged variables can be used to create orthogonal conditions. As a result, generalized method of moments (GMM) estimator or instrumental variables (IV) estimator can be adapted to estimate the parameters in the model. However, these linear transformation may not remove the time trend at the same time, and dynamic panel with both individual effects and time trend needs special treatment.

In this paper, we consider the forward demeaning (FOD) method of Hayakawa et al. (2017) and the double first difference (FD) (Wansbeek and Knaap (1999) and Hayakawa and Ge (2017)) to simultaneously remove the individual specific effects as well as the heterogenous trend in dynamic panels. Once the individual effects and the time trend are removed, we propose the Alvarez and Arellano (2003) type GMM estimator and simple IV estimator for the lag coefficient. It is shown in the paper that, the GMM estimator based on FOD is consistent and asymptotically normally distributed, but it is asymptotically biased of order $\sqrt{\frac{T}{N}}$, while for the GMM estimator based on FD is asymptotically normally distributed, but it is asymptotically biased of order $\sqrt{\frac{T^3}{N}}$. We also show that the simple IV estimator based on either FOD or FD transformed model is asymptotically unbiased and asymptotically normally distributed. The finite sample properties of both the GMM and IV estimation are examined through Monte Carlo simulations, and we find that the simulation results confirm our theoretical findings in the paper.

The rest of the paper is organized as follows. Section 2 sets up the basic model and discusses the approaches to eliminate both the individual effects and time trend. The GMM and simple IV estimation together with their associated asymptotics are provided in Section 3 and Section 4, respectively. Section 5 presents the finite sample properties of the GMM and IV estimation through Monte Carlo simulations, and Concluding remarks are in section 6. All mathematical derivations and additional simulation results are relegated to the Appendix.

2 Model

2.1 Model

We consider a simple dynamic panel model of the form

$$y_{it} = \gamma y_{i,t-1} + \alpha_i + \delta_i t + u_{it}, \quad i = 1, \ldots, N; t = 1, \ldots, T, \tag{2.1}$$

where $|\gamma| < 1$ and u_{it} has zero mean given $\alpha_i, \delta_i, y_{i0}, \ldots, y_{i,t-1}$. For ease of notations, we assume that y_{i0} is observed. This model has been widely used in empirical studies in economics, to name a few, see Hegwood and Papell (2007) and Conti (2014), among others.

For model (2.1), the goal is to estimate the lag coefficient γ with the observed y_{it} when both N and T are large. To this end, we make the following assumptions:

Assumption 1 (A1). $\{u_{it}\}$ are independent and identically distributed (i.i.d) across time and individuals and independent of y_{i0}, α_i and δ_i, with $E(u_{it}) = 0$, $Var(u_{it}) = \sigma_u^2$, and finite moments up to fourth order.

Assumption 2 (A2). Individual-specific effects α_i are i.i.d across individuals with $E(\alpha_i) = 0$, $Var(\alpha_i) = \sigma_\alpha^2$, and finite fourth order moment.

Assumption 3 (A3). The coefficient for the trend δ_i are i.i.d across individuals and independent of α_i, with $E(\delta_i) = 0$, $Var(\delta_i) = \sigma_\delta^2$, and finite fourth order moment.

Assumption 4 (A4). The initial observations y_{i0} satisfy

$$y_{i0} = \frac{\alpha_i}{1-\gamma} - \frac{\gamma\delta_i}{(1-\gamma)^2} + w_{i0},$$

where w_{i0} is independent of α_i and δ_i, and i.i.d with the steady state distribution of the homogeneous process, so that $w_{i0} = \sum_{s=0}^{\infty} \gamma^j u_{i,-s}$.

The above assumptions are quite standard in the literature for dynamic panel models, e.g., Alvarez and Arellano (2003), Hayakawa (2009). It should be noted that the homoskedasticity assumption in (A1) is for simplicity, all results obtained below can be extended to heteroskedastic errors with extra notations. The assumption of initial condition on y_{i0} is also made on Hayakawa and Nogimori (2010).

2.2 Elimination of Individual Effects and Time Trend

For model (2.1), stacking the observations over time period yields the vector form of

$$\mathbf{y}_i = \gamma \mathbf{y}_{i,-1} + \alpha_i 1_T + \delta_i \tau_T + \mathbf{u}_i, \; i = 1, \ldots, N, \tag{2.2}$$

where $\tau_T = (1, 2, \ldots, T)'$.

For the above model, we consider the following two types of transformation to simultaneously remove the individual-specific effects as well as the heterogeneous time trend. The first approach is the so called forward demeaning (FOD, for short)[1] (Hayakawa et al (2017)). More specifically, for model (2.2), let $T_j = T - j$ and define the following matrix

$$
\begin{aligned}
\mathbf{F}_{T_2 \times T} &= \mathbf{C}_{T_2} \left(\begin{array}{cc|cccc|cc}
1 & \frac{2(-2T_2)}{T_1T_2} & \frac{2(-2T_2+3)}{T_1T_2} & \frac{2(-2T_2+6)}{T_1T_2} & \cdots & \cdots & \frac{2(-2T_2+3T_3)}{T_1T_2} & \frac{2(-2T_2+3T_2)}{T_1T_2} \\
0 & 1 & \frac{2(-2T_3)}{T_2T_3} & \frac{2(-2T_3+3)}{T_2T_3} & \cdots & \cdots & \frac{2(-2T_3+3T_4)}{T_2T_3} & \frac{2(-2T_3+3T_3)}{T_2T_3} \\
\hline
0 & 0 & 1 & \frac{2(-2T_4)}{T_3T_4} & \cdots & \cdots & \frac{2(-2T_4+3T_5)}{T_3T_4} & \frac{2(-2T_4+3T_4)}{T_3T_4} \\
\vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 1 & \frac{2(-2\cdot 2)}{3\cdot 2} & \frac{2(-4+3)}{3\cdot 2} & \frac{2(-4+6)}{3\cdot 2} \\
0 & 0 & \cdots & \cdots & 0 & 1 & \frac{2(-2)}{2\cdot 1} & \frac{2(-2+3)}{2\cdot 1}
\end{array} \right) \\
&= \{f_{st}\} = \begin{cases} 0 & \text{if } s > t \\ c_t = 1 + O\left(\frac{1}{T-t}\right) & \text{if } s = t \\ \frac{2c_s[-2(T-s-1)+3(t-s-1)]}{(T-s)(T-s-1)} = -O\left(\frac{1}{T-s}\right) + O\left(\frac{t-s}{(T-s)^2}\right) & \text{if } s < t \end{cases}, \qquad (2.3)
\end{aligned}
$$

with $\mathbf{C}_{T_2} = diag\,(c_1, c_2, \ldots, c_{T-2})$ and $c_t^2 = \frac{(T-t-1)(T-t)}{(T-t+1)(T-t+2)}$. Then we have[2]

$$
\mathbf{F}_{T_2 \times T}\left(\alpha_i 1_T + \delta_i \tau_T\right) = 0, \qquad (2.4)
$$

and multiplying both sides of (2.2) by $\mathbf{F}_{T_2 \times T}$ yields

$$
\mathbf{y}_i^{(f)} = \gamma \mathbf{y}_{i,-1}^{(f)} + \mathbf{u}_i^{(f)},\ i = 1, \ldots, N, \qquad (2.5)
$$

where $\mathbf{y}_i^{(f)} = \mathbf{F}_{T_2 \times T}\mathbf{y}_i$, $\mathbf{y}_{i,-1}^{(f)} = \mathbf{F}_{T_2 \times T}\mathbf{y}_{i,-1}$ and $\mathbf{u}_i^{(f)} = \mathbf{F}_{T_2 \times T}\mathbf{u}_i$. The t-th element of the above transformed model is given by

$$
y_{it}^{(f)} = \gamma y_{it-1}^{(f)} + u_{it}^{(f)}, \quad i = 1, \ldots, N; t = 1, \ldots, T-2, \qquad (2.6)
$$

where

$$
\begin{aligned}
y_{it}^{(f)} &= f_{tt} y_{it} + f_{t,t+1} y_{i,t+1} + \cdots + f_{t,T} y_{iT}, \\
y_{i,t-1}^{(f)} &= f_{tt} y_{i,t-1} + f_{t,t+1} y_{i,t} + \cdots + f_{t,T} y_{i,T-1}, \\
u_{it}^{(f)} &= f_{tt} u_{it} + f_{t,t+1} u_{i,t+1} + \cdots + f_{t,T} u_{iT}.
\end{aligned}
$$

The second approach is the so called first difference (FD, for short) (Anderson and Hsiao (1981, 1982), Arellano and Bond (1991), Wansbeek and Knaap (1999)). More specifically, for

[1]It should be noted that the usual FOD (Alvarez and Arellano (2003), Arellano and Bover (1995)) can't remove the time trend in model (2.1).

[2]A formal derivation is given in the appendix.

model (2.1), the FD yields

$$\Delta y_{it} = \gamma \Delta y_{i,t-1} + \delta_i + \Delta u_{it}, \quad i = 1, \ldots, N; t = 2, \ldots, T, \tag{2.7}$$

with $\Delta y_{it} = y_{it} - y_{i,t-1}$, and another FD will eliminate δ_i from (2.7) as well

$$\Delta^2 y_{it} = \gamma \Delta^2 y_{i,t-1} + \Delta^2 u_{it}, \quad i = 1, \ldots, N; t = 3, \ldots, T, \tag{2.8}$$

where $\Delta^2 y_{it} = \Delta y_{it} - \Delta y_{it-1}$.

3 GMM Estimation and Their Asymptotics

In this section, we consider the GMM estimation of the lag coefficient γ based on either the FOD or FD approach to eliminate individual-specific effects and the heterogeneous time trend, and we also establish the asymptotics for the GMM estimation.

3.1 GMM Estimation Based on FOD

For model (2.6), under Assumptions (A1)-(A4), we have

$$E\left(y_{is} u_{it}^{(f)}\right) = 0, \text{ for any } s < t, \tag{3.1}$$

thus, we can use $(y_{i0}, \ldots, y_{it-1})$ as IVs for model (2.6).

We notice that model (2.6) can be rewritten in vector form as

$$\mathbf{y}_t^{(f)} = \gamma \mathbf{y}_{t-1}^{(f)} + \mathbf{u}_t^{(f)}, \quad t = 1, \ldots, T-2, \tag{3.2}$$

where $\mathbf{y}_t^{(f)} = \left(y_{1t}^{(f)}, y_{2t}^{(f)}, \ldots, y_{Nt}^{(f)}\right)'$, $\mathbf{y}_{t-1}^{(f)} = \left(y_{1,t-1}^{(f)}, y_{2,t-1}^{(f)}, \ldots, y_{N,t-1}^{(f)}\right)'$ and $\mathbf{u}_t^{(f)} = \left(u_{1t}^{(f)}, u_{2t}^{(f)}, \ldots, u_{Nt}^{(f)}\right)'$. Define $\mathbf{y}_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$ and $\mathbf{P}_{t-1} = \mathbf{Z}_{t-1}\left(\mathbf{Z}_{t-1}'\mathbf{Z}_{t-1}\right)^{-1}\mathbf{Z}_{t-1}'$ with $\mathbf{Z}_{t-1} = (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{t-1})$ for $t = 1, \ldots, T-2$, then the GMM of γ based on (3.2) using $\mathbf{Z}_{t-1}$ as the instruments is given by (Alvarez and Arellano (2003))

$$\hat{\gamma}_{GMM}^{FOD} = \left(\sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{y}_{t-1}^{(f)}\right)^{-1} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{y}_t^{(f)}. \tag{3.3}$$

3.2 GMM Estimation Based on FD

For the first differenced model (2.8), under Assumptions (A1)-(A4), we have the following orthogonal conditions,

$$E\left(y_{is} \Delta^2 u_{it}\right) = 0, \text{ for } s < t-2, \tag{3.4}$$

i.e., we can use $(y_{i0}, y_{i1}, \ldots, y_{i,t-3})$ as IV for (2.8).

It can also be noted that model (2.8) can be rewritten in vector form as

$$\Delta^2 \mathbf{y}_t = \gamma \Delta^2 \mathbf{y}_{t-1} + \Delta^2 \mathbf{u}_t,\ t = 3, \ldots, T; \tag{3.5}$$

where $\Delta^2 \mathbf{y}_t = \left(\Delta^2 y_{1t}, \Delta^2 y_{2t}, \ldots, \Delta^2 y_{Nt}\right)'$, $\Delta^2 \mathbf{y}_{t-1} = \left(\Delta^2 y_{1t-1}, \Delta^2 y_{2t-1}, \ldots, \Delta^2 y_{Nt-1}\right)'$ and $\Delta^2 \mathbf{u}_t = \left(\Delta^2 u_{1t}, \Delta^2 u_{2t}, \ldots, \Delta^2 u_{Nt}\right)'$, then the GMM estimation of γ based on (3.5) is given by (Alvarez and Arellano (2003))

$$\hat{\gamma}_{GMM}^{FD} = \left(\sum_{t=3}^{T} \Delta^2 \mathbf{y}_{t-1}' \mathbf{P}_{t-3} \Delta^2 \mathbf{y}_{t-1}\right)^{-1} \sum_{t=3}^{T} \Delta^2 \mathbf{y}_{t-1}' \mathbf{P}_{t-3} \Delta^2 \mathbf{y}_t, \tag{3.6}$$

where $\mathbf{P}_{t-3} = \mathbf{Z}_{t-3} \left(\mathbf{Z}_{t-3}' \mathbf{Z}_{t-3}\right)^{-1} \mathbf{Z}_{t-3}'$ with $\mathbf{Z}_{t-3} = (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{t-3})$.

3.3 Asymptotics of the GMM Estimator Based on FOD

For the GMM estimator (3.3), we note that

$$\sqrt{NT}\left(\hat{\gamma}_{GMM}^{FOD} - \gamma\right) = \left(\frac{1}{NT} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{y}_{t-1}^{(f)}\right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{u}_t^{(f)}, \tag{3.7}$$

and it is shown by Lemma A.3 in the appendix that

$$\frac{1}{NT} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{y}_{t-1}^{(f)} \to_p \frac{\sigma_u^2}{1-\gamma^2}, \tag{3.8}$$

and

$$\frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{u}_t^{(f)} \to_d N\left(-\frac{\sigma_u^2}{1-\gamma}\sqrt{\frac{T}{N}}, \frac{\sigma_u^4}{1-\gamma^2}\right). \tag{3.9}$$

under restriction $\frac{T}{N} \to c \neq 0 < \infty$ as $(N, T) \to \infty$.

Substituting (3.8) and (3.9) into (3.7) yields the limiting distribution of $\hat{\gamma}_{GMM}^{FOD}$, which is summarized in the following theorem.

Theorem 1 *For the GMM estimator (3.3), under Assumptions (A1)-(A4) and as* $(N, T) \to \infty$ *with* $\frac{T}{N} \to c \neq 0 < \infty$, *we have*

$$\sqrt{NT}\left(\hat{\gamma}_{GMM}^{FOD} - \gamma\right) \to_d N\left(-(1+\gamma)\sqrt{c}, 1-\gamma^2\right). \tag{3.10}$$

Remark 1 *From (3.10), we can observe that the GMM estimator (3.3) is consistent as long as* $N \to \infty$ *since*

$$\hat{\gamma}_{GMM}^{FOD} - \gamma = -\frac{(1+\gamma)}{N} + O_p\left((NT)^{-1/2}\right),$$

but it is asymptotically biased of order $\sqrt{\frac{T}{N}}$ *if* $\frac{T}{N} \to c \neq 0 < \infty$ *as* $(N,T) \to \infty$. *We can also note that the asymptotic distribution of the GMM estimator (3.3) for dynamic model with a trend is identical to the GMM estimator for dynamic model without a trend as in Alvarez and Arellano (2003). The intuition of this observation is that the instruments set for the GMM estimation of dynamic panel with or without a time trend are identical, so there will be no efficiency loss in the estimation.*

3.4 Asymptotics of the GMM Estimator based on FD

For the GMM estimator (3.6), we note that

$$\sqrt{NT}\left(\hat{\gamma}_{GMM}^{FD} - \gamma\right) = \left(\frac{1}{NT}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{y}_{t-1}\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t, \quad (3.11)$$

and it is shown by Lemma A.4 in the appendix that

$$\frac{1}{NT}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{y}_{t-1} \to_p \frac{(1-\gamma)^3}{1+\gamma}\sigma_u^2, \quad (3.12)$$

and

$$\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \to_d N\left(-\frac{(4-\gamma)\,\sigma_u^2}{2}\sqrt{\frac{T^3}{N}}, \frac{2\,(1-\gamma)^4\,(3-\gamma)\sigma_u^4}{1+\gamma}\right), \quad (3.13)$$

under the restriction that $\frac{T^3}{N} \to \kappa \neq 0 < \infty$ as $(N,T) \to \infty$.

Substituting (3.12) and (3.13) into (3.11) yields the limiting distribution of $\hat{\gamma}_{GMM}^{FD}$, which is summarized in the following theorem.

Theorem 2 *For the GMM estimator (3.6), under Assumptions (A1)-(A4) and as* $(N,T) \to \infty$ *with* $\frac{T^3}{N} \to \kappa \neq 0 < \infty$, *we have*

$$\sqrt{NT}\left(\hat{\gamma}_{GMM}^{FD} - \gamma\right) \to_d N\left(-\frac{(4-\gamma)\,(1+\gamma)}{2\,(1-\gamma)^3}\sqrt{\kappa}, \frac{2(1+\gamma)\,(3-\gamma)}{(1-\gamma)^2}\right), \quad (3.14)$$

Remark 2 *From (3.14), we can observe that the GMM estimator (3.6) is no long consistent if* $\frac{T}{N} \to c \neq 0 < \infty$ *as* $(N,T) \to \infty$ *since*

$$\hat{\gamma}_{GMM}^{FD} - \gamma = O_p\left(\frac{T}{N}\right),$$

and we need N *much larger than* T *to reach consistency. However, even if* $\frac{T}{N} \to 0$ *but* $\frac{T^3}{N} \to \kappa \neq 0 < \infty$ *as* $(N,T) \to \infty$, *the GMM estimator (3.6) is consistent but it is asymptotically biased of order* $\sqrt{\kappa}$. *It is also obvious that the asymptotic variance of (3.14) is much larger than that of (3.10), thus we can claim that (3.6) is not as efficient as (3.3).*

Remark 3 *Note that based on FOD transformation, there exist* $E\left(\Delta y_{is} u_{it}^{(f)}\right) = 0$ *for any* $s < t$*, and based on FD, we also have* $E\left(\Delta y_{is} \Delta^2 u_{it}\right) = 0$ *for any* $s < t-2$*. Consequently, in addition to using level lags as instruments, we can also use first difference lags as instruments for GMM estimators. These estimators will be investigated and compared through simulation in the appendix.*

4 Simple IV Estimation and Their Asymptotics

For model (2.6), under Assumptions (A1)-(A4), we have

$$E\left(\Delta y_{is} u_{it}^{(f)}\right) = 0, \text{ for any } s < t, \tag{4.1}$$

since $u_{it}^{(f)}$ is a combination of all future errors since time t, thus the lagged variables are legitimate instruments. As a result, a simple IV estimator of γ based on the FOD transformation is given by (Hsiao and Zhou (2017))

$$\hat{\gamma}_{IV}^{FOD} = \left(\sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{y}_{t-1}^{(f)}\right)^{-1} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{y}_{t}^{(f)}. \tag{4.2}$$

For the asymptotics of (4.2), we note that

$$\sqrt{NT}\left(\hat{\gamma}_{IV}^{FOD} - \gamma\right) = \left(\frac{1}{NT}\sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{y}_{t-1}^{(f)}\right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{u}_{t}^{(f)}, \tag{4.3}$$

and it is shown in Lemma A.5 in the appendix that as $(N,T) \to \infty$, we have

$$\frac{1}{NT}\sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{y}_{t-1}^{(f)} \to_p \frac{\sigma_u^2}{1+\gamma}, \tag{4.4}$$

and

$$\frac{1}{\sqrt{NT}}\sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{u}_{t}^{(f)} \to_d N\left(0, \sigma_u^2\left(\frac{\sigma_\delta^2}{(1-\gamma)^2} + \frac{2\sigma_u^2}{1+\gamma}\right)\right). \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3) yields the asymptotic distribution of (4.2) as follows.

Theorem 3 *Under Assumptions (A1)-(A4), as* $(N,T) \to \infty$*, for the simple IV estimator (4.2) of dynamic panels with heterogeneous trend using FOD to eliminate the individual effects and the trend, we have*

$$\sqrt{NT}\left(\hat{\gamma}_{IV}^{FOD} - \gamma\right) \to_d N\left(0, \frac{\sigma_\delta^2}{\sigma_u^2}\frac{(1+\gamma)^2}{(1-\gamma)^2} + 2(1+\gamma)\right). \tag{4.6}$$

Remark 4 *From the above limiting distribution, it is obvious that the simple IV estimator (4.2) based on FOD is asymptotically unbiased, and the asymptotic unbiasedness is independent of the way of how (N,T) go to infinity.*

For the first differenced model (2.8), under Assumptions (A1)-(A4), we have the following orthogonal conditions,

$$E\left(\Delta y_{is}\Delta^2 u_{it}\right)=0,\ \text{for } s<t-2, \tag{4.7}$$

i.e., we can use $(\Delta y_{i1},\ldots,\Delta y_{i,t-3})$ as IV for (2.8). Thus, a simple IV estimator of γ based on the FD transformation is given by (Anderson and Hsiao (1981, 1982), Hsiao and Zhou (2017))

$$\hat{\gamma}_{IV}^{FD}=\left(\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)^{-1}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_t. \tag{4.8}$$

Similarly, we have

$$\sqrt{NT}\left(\hat{\gamma}_{IV}^{FD}-\gamma\right)=\left(\frac{1}{NT}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t, \tag{4.9}$$

and it is shown in Lemma A.6 in the appendix that as $(N,T)\to\infty$, we have

$$\frac{1}{NT}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\to_p\frac{(1-\gamma)^2}{1+\gamma}\sigma_u^2, \tag{4.10}$$

and

$$\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\to_d N\left(0,\frac{2\left(\gamma^2-5\gamma+10\right)\sigma_u^4}{1+\gamma}\right). \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9) yields the asymptotic distribution of (4.8) as follows.

Theorem 4 *Under Assumptions (A1)-(A4), as $(N,T)\to\infty$, for the simple IV estimator (4.8) of dynamic panels with heterogeneous trend using FD to eliminate the individual effects and the trend, we have*

$$\sqrt{NT}\left(\hat{\gamma}_{IV}^{FD}-\gamma\right)\to_d N\left(0,\frac{2\left(1+\gamma\right)\left(\gamma^2-5\gamma+10\right)}{\left(1-\gamma\right)^4}\right). \tag{4.12}$$

Remark 5 *Similar to the FOD case, the Anderson-Hsiao simple IV estimator (4.8) based on FD is asymptotically unbiased, and the asymptotic unbiasedness is independent of the way of how (N,T) go to infinity. It is also of interest to compare the efficiency of these two estimators, from (4.6) and (4.12), the ratio of these two asymptotic variance is given by*

$$\frac{\frac{\sigma_\delta^2}{\sigma_u^2}\frac{(1+\gamma)^2}{(1-\gamma)^2}+2\left(1+\gamma\right)}{\frac{2(1+\gamma)(\gamma^2-5\gamma+10)}{(1-\gamma)^4}}=\frac{\frac{\sigma_\delta^2}{\sigma_u^2}\left(1+\gamma\right)\left(1-\gamma\right)^2+2\left(1-\gamma\right)^4}{2\left(\gamma^2-5\gamma+10\right)},$$

as a result, one condition that the simple IV based on FOD is more efficient than the IV based on FD is that

$$\frac{\sigma_\delta^2}{\sigma_u^2} < \frac{2\left(\gamma^2 - 5\gamma + 10\right) - 2\left(1-\gamma\right)^4}{\left(1+\gamma\right)\left(1-\gamma\right)^2}.$$

Remark 6 *We notice that level lagged variables y_{is} also satisfy the orthogonal condition for (2.6) and (2.8), however, it is shown in the appendix that the simple IV estimation using one level lag for both FOD and FD model is invalid, and the simulation results confirm our theoretical findings.*

5 Monte Carlo Simulation

In this section, we investigate the finite sample properties for GMM estimation and simple IV estimation for dynamic panels with individual effects and time trend using either FOD or FD transformation discussed in the previous sections. We consider the following data generating process (DGP)

$$y_{it} = \gamma y_{i,t-1} + \alpha_i + \delta_i t + u_{it}, \tag{5.1}$$

we assume that $\alpha_i \sim IIDN\left(0,1\right), \delta_i \sim IIDU\left(-1,1\right)$, and $u_{it} \sim IIDN\left(0,1\right)$ for all i and t. For the values of γ, we let $\gamma = 0.2, 0.5$ and 0.7. We consider the combinations of $N = 200, 500, 800$ and $T = 25, 50, 100$. The number of replication is set at 2000 times.

For the above DGP, we use both the FOD and FD transformations to eliminate the individual effects, α_i, and time trend, $\delta_i t$. We consider two types of GMM estimators for estimating the lag coefficient γ: GMM based FOD and GMM based on FD using all available level lags, as well as two types of simple IV estimators using only one first differenced lag. We calculate median of estimates, median-bias as well as iqr (inter-quantile rage) for these estimators. The simulation results are summarized in Table 1-3.[3]

Several important findings can be observed from the simulation results. First, we find that the GMM estimator based on FOD and using all level lags behaves quite well, the median bias is almost negligible when N or T is large, and the iqr reduces quite rapidly with the increase of either N or T. We can also observe that the iqr for the GMM estimation based FOD is quite close to those obtained in Alvarez and Arellano (2003, P1134-1135). Second, the GMM based on FD using all level lags shows quite significant bias, but the bias decreases with the decrease

[3]Additional simulation results of the GMM using first differenced lagges as IVs, and simple IV using level lags as IV are provided in the appendix.

of the magnitude of $\frac{T}{N}$ (i.e., N increases faster than T). Finally, for simple IV estimation based on both FOD and FD using only one first differenced lag as IV, the bias is almost negligible, and the iqr also decreases with the increase of either N or T, which is the evident that the simple IV is asymptotically unbiased and consistent. In all, we conclude that the findings in the simulation confirm our theoretical findings in the paper.

6 Conclusion

In this paper, we consider both the FOD and FD transformations to eliminate the individual specific effects as well as the stochastic time trend in dynamic panels. We also consider the GMM and simple IV estimation based on either FOD or FD transformation using either all available instruments or only one instrument. Asymptotics for both GMM and simple IV estimation are established in the paper. Monte Carlo simulations confirm our theoretical findings in the paper.

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Table 1: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.2$

T	N	200				500				800			
		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.1744	-0.2414	0.1994	0.1994	0.1881	-0.0568	0.1993	0.1978	0.1929	0.0191	0.1998	0.1990
	bias	-0.0256	-0.4414	-0.0006	-0.0006	-0.0119	-0.2568	-0.0007	-0.0022	-0.0071	-0.1809	-0.0002	-0.0010
	iqr	0.0318	0.0572	0.0498	0.1406	0.0222	0.0440	0.0320	0.0891	0.0176	0.0376	0.0247	0.0700
	median	0.1816	-0.3557	0.2000	0.2000	0.1916	-0.1585	0.1998	0.2008	0.1948	-0.0648	0.1997	0.2013
50	bias	-0.0184	-0.5557	0.0000	0.0000	-0.0084	-0.3585	-0.0002	0.0008	-0.0052	-0.2648	-0.0003	0.0013
	iqr	0.0180	0.0276	0.0312	0.1042	0.0124	0.0263	0.0186	0.0654	0.0091	0.0232	0.0145	0.0499
	median	0.1852	-0.4662	0.2003	0.1996	0.1936	-0.2856	0.2000	0.2010	0.1959	-0.1821	0.1998	0.2014
100	bias	-0.0148	-0.6662	0.0003	-0.0004	-0.0064	-0.4856	0.0000	0.0010	-0.0041	-0.3821	-0.0002	0.0014
	iqr	0.0107	0.0142	0.0189	0.0685	0.0072	0.0139	0.0119	0.0446	0.0059	0.0127	0.0100	0.0350

Note: 1. "GMM" refers to the GMM estimation using all level lags as instruments, and "IV" refers to simple instrumental variable estimation using only one first differenced lag variable as instrument.

2. "FOD" refers to forward demeaning, and "FD" refers to double first difference.

3. iqr refers inter quantile range (75%-25%).

Table 2: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.5$

T	N	200				500				800			
		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.4373	-0.4103	0.4991	0.5005	0.4688	-0.2532	0.4998	0.4929	0.4792	-0.1413	0.5009	0.4989
	bias	-0.0627	-0.9103	-0.0009	0.0005	-0.0312	-0.7532	-0.0002	-0.0071	-0.0208	-0.6413	0.0009	-0.0011
	iqr	0.0431	0.0616	0.0876	0.3913	0.0314	0.0662	0.0567	0.2396	0.0255	0.0643	0.0445	0.1886
	median	0.4665	-0.4563	0.5004	0.5001	0.4839	-0.3264	0.4998	0.5000	0.4899	-0.2283	0.5002	0.5028
50	bias	-0.0335	-0.9563	0.0004	0.0001	-0.0161	-0.8264	-0.0002	0.0000	-0.0101	-0.7283	0.0002	0.0028
	iqr	0.0195	0.0285	0.0488	0.2809	0.0142	0.0314	0.0295	0.1733	0.0119	0.0309	0.0247	0.1340
	median	0.4777	-0.4992	0.5009	0.4991	0.4897	0.4067	0.4999	0.5011	0.4934	-0.3306	0.4999	0.5017
100	bias	-0.0223	-0.9992	0.0009	-0.0009	-0.0103	-0.9067	-0.0001	0.0011	-0.0066	-0.8306	-0.0001	0.0017
	iqr	0.0109	0.0143	0.0270	0.1832	0.0074	0.0142	0.0180	0.1159	0.0059	0.0151	0.0143	0.0920

See notes of Table 1

Table 3: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.7$

	N	200				500				800			
T		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.5605	-0.4925	0.7031	0.6408	0.6172	-0.4538	0.7028	0.7021	0.6413	-0.4208	0.7004	0.6986
	bias	-0.1395	-1.1925	0.0031	-0.0592	-0.0828	-1.1538	0.0028	0.0021	-0.0587	-1.1208	0.0004	-0.0014
	iqr	0.0610	0.0560	0.1948	1.1697	0.0481	0.0584	0.1343	0.7620	0.0409	0.0600	0.1031	0.5560
	median	0.6377	-0.5001	0.6989	0.6970	0.6665	-0.4672	0.7008	0.7026	0.6773	-0.4362	0.7004	0.7074
50	bias	-0.0623	-1.2001	-0.0011	0.0030	-0.0335	-1.1672	0.0008	0.0026	-0.0227	-1.1362	0.0004	0.0074
	iqr	0.0241	0.0257	0.1063	0.8044	0.0176	0.0284	0.0600	0.4994	0.0149	0.0292	0.0476	0.3948
	median	0.6675	-0.5080	0.7015	0.7025	0.6834	-0.4848	0.7003	0.6999	0.6890	-0.4618	0.7003	0.7014
100	bias	-0.0325	-1.2080	0.0015	0.0025	-0.0166	-1.1848	0.0003	-0.0001	-0.0110	-1.1618	0.0003	0.0014
	iqr	0.0115	0.0135	0.0481	0.5170	0.0076	0.0137	0.0318	0.3306	0.0061	0.0139	0.0258	0.2629

See notes of Table 1

Appendix

This Appendix includes the mathematical proofs and additional simulation results that are omitted in the main paper.

A Derivation of (2.4)

To show how the transformation matrix $\mathbf{F}_{T_2\times T}$ removes the individual effects as well as the time trend, let's look at the multiplication of the t-th row of $\mathbf{F}_{T_2\times T}$ with $(\alpha_i 1_T+\delta_i\tau_T)$, which is given by

$$\begin{aligned}
& (f_{t1},f_{t2},\ldots,f_{tT})(\alpha_i 1_T+\delta_i\tau_T) \\
= \ & c_t\alpha_i\left(1+2\sum_{s=t+1}^{T}\frac{[-2(T-t-1)+3(s-t-1)]}{(T-t)(T-t-1)}\right) \\
& +c_t\delta_i\left(t+2\sum_{s=t+1}^{T}\frac{[-2(T-t-1)+3(s-t-1)]}{(T-t)(T-t-1)}s\right),
\end{aligned} \tag{A.1}$$

where the first term of (A.1) can be shown

$$\begin{aligned}
\sum_{s=t+1}^{T}\frac{[-2(T-t-1)+3(s-t-1)]}{(T-t)(T-t-1)} &= \sum_{s=t+1}^{T}\frac{-2(T-t-1)}{(T-t)(T-t-1)}+\frac{3\sum_{s=t+1}^{T}(s-t-1)}{(T-t)(T-t-1)} \\
&= -2+\frac{3}{(T-t)(T-t-1)}\frac{(T-t-1)(T-t)}{2} \\
&= -\frac{1}{2},
\end{aligned}$$

which in turn gives

$$1+2\sum_{s=t+1}^{T}\frac{[-2(T-t-1)+3(s-t-1)]}{(T-t)(T-t-1)}=0.$$

Also, for the second term of (A.1), we have

$$\begin{aligned}
& \sum_{s=t+1}^{T}\frac{[-2(T-t-1)+3(s-t-1)]}{(T-t)(T-t-1)}s=\sum_{s=t+1}^{T}\frac{[-2T-t-1+3s]}{(T-t)(T-t-1)}s \\
= \ & \frac{(-2T-t-1)}{(T-t)(T-t-1)}\frac{(T-t)(T+t+1)}{2}+\frac{T(T+1)(2T+1)-t(t+1)(2t+1)}{2(T-t)(T-t-1)} \\
= \ & \frac{T(T+1)(2T+1)-t(t+1)(2t+1)-(2T+t+1)(T-t)(T+t+1)}{2(T-t)(T-t-1)} \\
= \ & \frac{-t^3-t^2+2Tt^2+2Tt-tT^2-tT}{2(T-t)(T-t-1)} \\
= \ & -t\frac{(T-t)^2+t-T}{2(T-t)(T-t-1)}=-\frac{t}{2},
\end{aligned}$$

which in turn yields

$$t+2\sum_{s=t+1}^{T}\frac{\left[-2\left(T-t-1\right)+3\left(s-t-1\right)\right]}{\left(T-t\right)\left(T-t-1\right)}s=0.$$

As a result, the transformation matrix $\mathbf{F}_{T_2\times T}$ indeed removes the individual-specific effects and the time trend of model (2.2) simultaneously.

B Derivation of the Asymptotics of the GMM and IV Estimation

Let $\|\mathbf{A}\|=\sqrt{tr\left(\mathbf{A}\mathbf{A}'\right)}$ denote the Frobenius norm, $\lambda_{\min}\left(\mathbf{A}\right)$ to denote the minimum eigenvalue of $\mathbf{A}$, and $\lambda_{\max}\left(\mathbf{A}\right)$ to denote the maximum eigenvalue of $\mathbf{A}$. We also let C denote a generic finite constant which doesn't depend on N or T, and whose value may vary case by case.

For model (2.1), under strict stationarity of y_{it}, we have

$$y_{it}\left(1-\gamma L\right)=\alpha_i+\delta_i t+u_{it},$$

or

$$y_{it}=\frac{\alpha_i}{1-\gamma}+\delta_i\frac{t}{1-\gamma L}+\frac{u_{it}}{1-\gamma L},$$

where it can be shown that $\frac{t}{1-\gamma L}=\frac{t}{1-\gamma}-\frac{\gamma}{(1-\gamma)^2}$, i.e.,

$$y_{it}=\alpha_i^*+\delta_i^* t+w_{it}, \tag{A.1}$$

where $\alpha_i^*=\frac{\alpha_i}{1-\gamma}-\delta_i\frac{\gamma}{(1-\gamma)^2}$, $\delta_i^*=\frac{\delta_i}{1-\gamma}$ and w_{it} is a AR(1) process with[4]

$$w_{it}=\gamma w_{it-1}+u_{it}=\sum_{s=0}^{\infty}\gamma^s u_{it-s}. \tag{A.2}$$

[4]By definition, it can be verified that

$$w_{it}=y_{it}-\frac{\alpha_i}{1-\gamma}+\delta_i\frac{\gamma}{\left(1-\gamma\right)^2}-\frac{\delta_i}{1-\gamma}t,$$

and

$$w_{it-1}=y_{it-1}-\frac{\alpha_i}{1-\gamma}+\delta_i\frac{\gamma}{\left(1-\gamma\right)^2}-\frac{\delta_i}{1-\gamma}\left(t-1\right),$$

then by substituting the above two terms into

$$w_{it}=\gamma w_{it-1}+u_{it},$$

which in turn yields

$$y_{it}=\alpha_i+\delta_i t+\gamma y_{it-1}+u_{it}.$$

It is obvious that for the FD approach to remove the individual effects as well as the trend, we have

$$\Delta^2 y_{it} = \Delta^2 w_{it}, \tag{A.3}$$

and for the FOD transformation, we have

$$y_{it}^{(f)} = w_{it}^{(f)}, \tag{A.4}$$

where

$$w_{i,t}^{(f)} = f_{tt} w_{it} + f_{t,t+1} w_{i,t+1} + \cdots + f_{t,T} w_{iT},$$

with f_{ts} $(s \geq t)$ is given by (2.3).

Now let's turn to the lemmas, which are needed for the derivation of the results in the main paper.

Lemma A.1 *Let $\kappa_{3,u}$ and $\kappa_{4,u}$ be the third and fourth order cumulants of u_{it}. Also, let $\mathbf{d}_t$ and $\mathbf{d}_s$ be $N \times 1$ vectors containing the diagonal elements of $\mathbf{P}_t = \mathbf{Z}_t \left(\mathbf{Z}_t'\mathbf{Z}_t\right)^{-1} \mathbf{Z}_t'$ and $\mathbf{P}_s = \mathbf{Z}_s \left(\mathbf{Z}_s'\mathbf{Z}_s\right)^{-1} \mathbf{Z}_s'$, respectively, so that $tr(\mathbf{P}_t) = \mathbf{d}_t' 1_N = t$ and $tr(\mathbf{P}_s) = \mathbf{d}_s' 1_N = s$, and $\mathbf{d}_t'\mathbf{d}_s \leq \min(t,s)$, then under Assumption (A1), for $l \geq r > t$, $p \geq q > s$ and $t \geq s$*

$$Cov\left(\mathbf{u}_l'\mathbf{P}_t\mathbf{u}_r, \mathbf{u}_p'\mathbf{P}_s\mathbf{u}_q\right) = \begin{cases} \kappa_{4,u}\mathbf{d}_t'\mathbf{d}_s + 2\sigma_u^4 s \leq \left(\kappa_{4,u} + 2\sigma_u^4\right) s & \textit{if } l = r = p = q \\ \sigma_u^4 s & \textit{if } l = p \neq r = q \\ \kappa_{3,u} E\left(\mathbf{d}_t'\mathbf{p}_s\mathbf{u}_q\right) & \textit{if } l = r = p \neq q < t \\ 0 & \textit{otherwise} \end{cases}$$

and $\left|E\left(\mathbf{d}_t'\mathbf{p}_s\mathbf{u}_q\right)\right| \leq (st)^{1/2}\sigma_u$.

Proof can be found at Alvarez and Arellano (2003). The above result holds for models with trend is because the trace and rank of $\mathbf{P}_t$ and $\mathbf{P}_s$ remain the same for models without trend, so the derivation of Alvarez and Arellano (2003) can be applied here.

Lemma A.2 *Let Assumptions (A1)-(A4) hold, then the following holds as $(N,T) \to \infty$,*

$$\frac{1}{NT}\sum_{t=1}^{T-2} \mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1} \to_p \frac{\sigma_u^2}{1-\gamma^2}.$$

Proof. To derive this result, similar to Lee et al. (2017), we notice that

$$\frac{1}{NT}\sum_{t=1}^{T-2} \mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1} = \frac{1}{NT}\sum_{t=1}^{T-2} \mathbf{w}_{t-1}'\mathbf{w}_{t-1} - \frac{1}{NT}\sum_{t=1}^{T-2} \mathbf{w}_{t-1}'\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\mathbf{w}_{t-1}, \tag{A.5}$$

and $\mathbf{w}_{t-1} = \mathbf{y}_{t-1} - \boldsymbol{\alpha}^* - \boldsymbol{\delta}^*(t-1)$ where $\boldsymbol{\alpha}^* = \left(\alpha_1^*, \ldots, \alpha_N^*\right)'$ with $\alpha_i^* = \frac{\alpha_i}{1-\gamma} - \delta_i \frac{\gamma}{(1-\gamma)^2}$ and $\boldsymbol{\delta}^* = \left(\delta_1^*, \ldots, \delta_N^*\right)'$ with $\delta_i^* = \frac{\delta_i}{1-\gamma}$.

Let $\boldsymbol{\mu}_{t-1}$ be the $N \times 1$ vector of errors of the population linear projection of $\boldsymbol{\alpha}^*$ on $\mathbf{Z}_{t-1}$, then we have

$$\boldsymbol{\mu}_{t-1} = \boldsymbol{\alpha}^* - \mathbf{Z}_{t-1}\mathbf{a}_{t-1} \quad \text{with } \mathbf{a}_{t-1} = \left[E\left(\mathbf{z}_{it-1}\mathbf{z}'_{it-1}\right)\right]^{-1} E\left(\mathbf{z}_{it-1}\alpha_i^*\right). \tag{A.6}$$

Similarly, let $\boldsymbol{\epsilon}_{t-1}$ be the $N \times 1$ vector of errors of the population linear projection of $\boldsymbol{\delta}^*$ on $\mathbf{Z}_{t-1}$, then we have

$$\boldsymbol{\epsilon}_{t-1} = \boldsymbol{\delta}^* - \mathbf{Z}_{t-1}\mathbf{b}_{t-1} \quad \text{with } \mathbf{b}_{t-1} = \left[E\left(\mathbf{z}_{it-1}\mathbf{z}'_{it-1}\right)\right]^{-1} E\left(\mathbf{z}_{it-1}\delta_i^*\right), \tag{A.7}$$

where $\mathbf{z}_{it-1} = \left(y_{i0}, \ldots, y_{i,t-1}\right)'$.

In order to derive the properties of $\boldsymbol{\mu}_{t-1}$ and $\boldsymbol{\epsilon}_{t-1}$, by using the decomposition, we have

$$\begin{aligned} w_{it} &= \gamma w_{it-1} + u_{it}, \\ y_{it} &= \alpha_i^* + \delta_i^* t + w_{it}, \\ \mathbf{z}_{it-1} &= \alpha_i^* \mathbf{1}_t + \delta_i^* \boldsymbol{\tau}_{t-1} + \mathbf{w}_i^{(t-1)}, \end{aligned}$$

where $\boldsymbol{\tau}_{t-1} = (0, 1, \ldots, t-1)'$ and $\mathbf{w}_i^{(t-1)} = \left(w_{i0}, \ldots, w_{i,t-1}\right)'$ with $w_{i,t-1}$ is an AR(1) process. Let $\boldsymbol{\Omega}_t$ be the $t \times t$ autoregressive matrix of w_{it} whose (j,k)-th element is given by $\frac{\gamma^{|j-k|}}{1-\gamma^2}$ for $j, k = 0, 1, \ldots, t-1$, then we have

$$\begin{aligned} E\left(\mathbf{z}_{it-1}\mathbf{z}'_{it-1}\right) &= \sigma^2_{\alpha^*}\mathbf{1}_t\mathbf{1}'_t + \sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t, \\ E\left(\mathbf{z}_{it-1}\alpha_i^*\right) &= \sigma^2_{\alpha^*}\mathbf{1}_t, \\ E\left(\mathbf{z}_{it-1}\delta_i^*\right) &= \sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}, \end{aligned}$$

where $\sigma^2_{\alpha^*} = E\left(\alpha_i^{*2}\right)$ and $\sigma^2_{\delta^*} = E\left(\delta_i^{*2}\right)$.

To get an exact expression of the inverse of $E\left(\mathbf{z}_{it-1}\mathbf{z}'_{it-1}\right)$, we notice that

$$\begin{aligned} & E\left(\mathbf{z}_{it-1}\mathbf{z}'_{it-1}\right)^{-1} \\ = & \left(\sigma^2_{\alpha^*}\mathbf{1}_t\mathbf{1}'_t + \sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1} \\ = & \left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1} - \frac{\sigma^2_{\alpha^*}\left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t\mathbf{1}'_t\left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}}{1 + \sigma^2_{\alpha^*}\mathbf{1}'_t\left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t}, \end{aligned}$$

and

$$\left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1} = \sigma_u^{-2}\left(\varkappa_1\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \boldsymbol{\Omega}_t\right)^{-1} = \sigma_u^{-2}\left[\boldsymbol{\Omega}_t^{-1} - \frac{\varkappa_1\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1}\boldsymbol{\Omega}_t^{-1}}{1 + \varkappa_1\boldsymbol{\tau}'_{t-1}\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}}\right], \tag{A.8}$$

where $\varkappa_1 = \frac{\sigma^2_{\delta^*}}{\sigma^2_u}$. Then we have

$$\mathbf{1}'_t\left(\sigma^2_{\delta^*}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}'_{t-1} + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t = \sigma_u^{-2}\mathbf{1}'_t\boldsymbol{\Omega}_t^{-1}\mathbf{1}_t - \frac{\sigma_u^{-2}\varkappa_1\left(\mathbf{1}'_t\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}\right)^2}{1 + \varkappa_1\boldsymbol{\tau}'_{t-1}\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}} = O(t), \tag{A.9}$$

since it can be shown that

$$\mathbf{\Omega}_t = \frac{1}{1-\gamma^2}\begin{pmatrix} 1 & \gamma & \cdots & \gamma^{t-1} \\ \gamma & 1 & \cdots & \gamma^{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{t-1} & \gamma^{t-2} & \cdots & 1 \end{pmatrix}, \text{ and } \mathbf{\Omega}_t^{-1} = \begin{pmatrix} 1 & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+\gamma^2 & -\gamma & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\gamma & 1+\gamma^2 & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1 \end{pmatrix},$$

which in turn gives

$$\begin{aligned} \mathbf{\Omega}_t^{-1}\mathbf{1}_t &= (1-\gamma)(1, 1-\gamma, \ldots, 1-\gamma, 1)', \\ \mathbf{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1} &= \left(-\gamma, (1-\gamma)^2, 2(1-\gamma)^2 \ldots, (t-2)(1-\gamma)^2, (t-1)(1-\gamma)+\gamma\right)', \end{aligned}$$

then it is obvious that

$$\boldsymbol{\tau}_{t-1}'\mathbf{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1} = O\left(t^3\right) \text{ and } \mathbf{1}_t'\mathbf{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1} = O\left(t^2\right). \tag{A.10}$$

As a result, we have

$$\begin{aligned} \mathbf{a}_{t-1} &= \left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\alpha^*}^2\mathbf{1}_t \\ &= \left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\alpha^*}^2\mathbf{1}_t \\ &\quad - \frac{\sigma_{\alpha^*}^4\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t}{1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t} \\ &= \left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\alpha^*}^2\mathbf{1}_t \times \left(1 - \frac{\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t}{1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t}\right) \\ &= \frac{\sigma_{\alpha^*}^2\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t}{1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\mathbf{1}_t}, \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} \mathbf{b}_{t-1} &= \left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}' + \sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1} \\ &= \left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1} \\ &\quad - \frac{\sigma_{\delta^*}^2\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}}{1+\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}} \\ &= \frac{\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}}{1+\sigma_{\alpha^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\mathbf{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}}. \end{aligned} \tag{A.12}$$

Consequently, given the above two coefficients, we have, for the ith element of $\boldsymbol{\mu}_{t-1}$,

$$\mu_{it-1} = \alpha_i^* - \mathbf{z}_{it-1}'\mathbf{a}_{t-1} = \alpha_i^*\left(1 - \mathbf{1}_t'\mathbf{a}_{t-1}\right) - \delta_i^*\boldsymbol{\tau}_{t-1}'\mathbf{a}_{t-1} - \mathbf{w}_i^{(t-1)\prime}\mathbf{a}_{t-1}.$$

For $E\left(\mu_{it-1}^2\right)$, under Assumptions (A1)-(A3), we have

$$E\left(\mu_{it-1}^2\right)=\sigma_{\alpha^*}^2\left(1-\mathbf{1}_t'\mathbf{a}_{t-1}\right)^2+\sigma_{\delta^*}^2\left(\boldsymbol{\tau}_{t-1}'\mathbf{a}_{t-1}\right)^2+\mathbf{a}_{t-1}'E\left(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime}\right)\mathbf{a}_{t-1}, \tag{A.13}$$

where $E\left(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime}\right)=\sigma_u^2\boldsymbol{\Omega}_t$.

For the first term, we notice that

$$\begin{aligned}1-\mathbf{1}_t'\mathbf{a}_{t-1} &= 1-\frac{\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t}{1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t}\\ &= \frac{1}{1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t}\\ &= O\left(\frac{1}{t}\right),\end{aligned}$$

by using the result (A.11), consequently, we have

$$\sigma_{\alpha^*}^2\left(1-\mathbf{1}_t'\mathbf{a}_{t-1}\right)^2=O\left(\frac{1}{t^2}\right). \tag{A.14}$$

For the second term of (A.13),

$$\sigma_{\delta^*}^2\left(\boldsymbol{\tau}_{t-1}'\mathbf{a}_{t-1}\right)^2=O\left(\frac{1}{t^4}\right), \tag{A.15}$$

since

$$\begin{aligned}\boldsymbol{\tau}_{t-1}'\mathbf{a}_{t-1} &= \frac{\sigma_{\alpha^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\Omega_t\right)^{-1}1_t}{1+\sigma_{\alpha^*}^2 1_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\Omega_t\right)^{-1}1_t}\\ &= \frac{\sigma_{\alpha^*}^2\sigma_u^{-2}1_t'\Omega_t^{-1}\boldsymbol{\tau}_{t-1}}{\left(1+\varkappa_1\boldsymbol{\tau}_{t-1}'\Omega_t^{-1}\boldsymbol{\tau}_{t-1}\right)\left(1+\sigma_{\alpha^*}^2 1_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\Omega_t\right)^{-1}1_t\right)}=O\left(\frac{1}{t^2}\right),\end{aligned}$$

by using the results (A.8)-(A.10).

Finally, for the third term of (A.13), we have

$$\begin{aligned}\mathbf{a}_{t-1}'E\left(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime}\right)\mathbf{a}_{t-1} &= \frac{\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\sigma_u^2\boldsymbol{\Omega}_t\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t'}{\left(1+\sigma_{\alpha^*}^2\mathbf{1}_t'\left(\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}\boldsymbol{\tau}_{t-1}'+\sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\mathbf{1}_t\right)^2}\\ &= O\left(\frac{1}{t}\right),\end{aligned} \tag{A.16}$$

by using the similar argument above.

As a result, combining (A.14)-(A.16) we have

$$E\left(\mu_{it-1}^2\right)=O\left(\frac{1}{t}\right). \tag{A.17}$$

Similarly, for the residuals of projection on $\boldsymbol{\delta}^*$, we have, for the ith element of $\boldsymbol{\epsilon}_{t-1}$,

$$\epsilon_{it-1} = \delta_i^* - \mathbf{z}_{it-1}'\mathbf{b}_{t-1} = -\alpha_i^*\mathbf{1}_t'\mathbf{b}_{t-1} + \delta_i^*\left(1 - \boldsymbol{\tau}_{t-1}'\mathbf{b}_{t-1}\right) - \mathbf{w}_i^{(t-1)\prime}\mathbf{b}_{t-1}.$$

For $E\left(\epsilon_{it-1}^2\right)$, we note

$$E\left(\epsilon_{it-1}^2\right) = \sigma_{\alpha^*}^2\left(\mathbf{1}_t'\mathbf{b}_{t-1}\right)^2 + \sigma_{\delta^*}^2\left(1 - \boldsymbol{\tau}_{t-1}'\mathbf{b}_{t-1}\right)^2 + \mathbf{b}_{t-1}'E\left(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime}\right)\mathbf{b}_{t-1}. \tag{A.18}$$

For the first term $\sigma_{\alpha^*}^2\left(\mathbf{1}_t'\mathbf{b}_{t-1}\right)^2$, we notice that

$$\begin{aligned}
\mathbf{1}_t'\mathbf{b}_{t-1} &= \frac{\mathbf{1}_t'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}}{1 + \sigma_{\alpha^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}} \\
&= \frac{\sigma_{\delta^*}^2\sigma_u^{-2}\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}}{\left(1 + \sigma_{\alpha^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}\right)\left(1 + \varkappa_2\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\mathbf{1}_t\right)} \\
&= O\left(\frac{1}{t^2}\right),
\end{aligned}$$

by using the identity

$$\mathbf{1}_t'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1} = \sigma_u^{-2}\mathbf{1}_t'\Omega_t^{-1}\boldsymbol{\tau}_{t-1} - \frac{\sigma_u^{-2}\varkappa_2\mathbf{1}_t'\Omega_t^{-1}\mathbf{1}_t\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}}{1 + \varkappa_2\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\mathbf{1}_t} = \sigma_u^{-2}\frac{\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}}{1 + \varkappa_2\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\mathbf{1}_t},$$

and

$$\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1} = \sigma_u^{-2}\boldsymbol{\tau}_{t-1}'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1} - \frac{\sigma_u^{-2}\varkappa_2\left(\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\tau}_{t-1}\right)^2}{1 + \varkappa_2\mathbf{1}_t'\boldsymbol{\Omega}_t^{-1}\mathbf{1}_t} = O\left(t^3\right),$$

where $\varkappa_2 = \frac{\sigma_{\alpha^*}^2}{\sigma_u^2}$. Consequently, we have

$$\sigma_{\alpha^*}^2\left(\mathbf{1}_t'\mathbf{b}_{t-1}\right)^2 = O\left(\frac{1}{t^4}\right). \tag{A.19}$$

For the second term of (A.18), we notice that

$$\begin{aligned}
1 - \boldsymbol{\tau}_{t-1}'\mathbf{b}_{t-1} &= 1 - \frac{\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}}{1 + \sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}} \\
&= \frac{1}{1 + \sigma_{\delta^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}} \\
&= O\left(\frac{1}{t^3}\right).
\end{aligned} \tag{A.20}$$

Finally, for the third term of (A.18), we have

$$\begin{aligned}
\mathbf{b}_{t-1}' E\left(\mathbf{w}_i^{(t-1)}\mathbf{w}_i^{(t-1)\prime}\right)\mathbf{b}_{t-1} &= \mathbf{b}_{t-1}'\sigma_u^2\boldsymbol{\Omega}_t\mathbf{b}_{t-1} \leq C\mathbf{b}_{t-1}'\mathbf{b}_{t-1} \\
&\leq C\frac{\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}}{\left(1+\sigma_{\alpha^*}^2\boldsymbol{\tau}_{t-1}'\left(\sigma_{\alpha^*}^2\mathbf{1}_t\mathbf{1}_t' + \sigma_u^2\boldsymbol{\Omega}_t\right)^{-1}\boldsymbol{\tau}_{t-1}\right)^2} \\
&= O\left(\frac{1}{t^3}\right). \qquad (A.21)
\end{aligned}$$

As a result, combining (A.19)-(A.21) we have

$$E\left(\epsilon_{it-1}^2\right) = O\left(\frac{1}{t^3}\right). \qquad (A.22)$$

Now let's turn to equation (A.5), by definition of (A.6) and (A.7),

$$\mathbf{w}_{t-1} = \mathbf{y}_{t-1} - \boldsymbol{\alpha}^* - \boldsymbol{\delta}^*(t-1) = \mathbf{y}_{t-1} - \left(\mathbf{Z}_{t-1}\mathbf{a}_{t-1} + \boldsymbol{\mu}_{t-1}\right) - \left(\mathbf{Z}_{t-1}\mathbf{b}_{t-1} + \boldsymbol{\epsilon}_{t-1}\right)(t-1),$$

then the second term of (A.5) will reduce to

$$\begin{aligned}
& \frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\mathbf{w}_{t-1} \\
= & \frac{1}{NT}\sum_{t=1}^{T-2}\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right)'\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right) \\
\leq & \frac{1}{NT}\sum_{t=1}^{T-2}\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right)'\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right) \\
= & \frac{1}{NT}\sum_{t=1}^{T-2}\boldsymbol{\mu}_{t-1}'\boldsymbol{\mu}_{t-1} + \frac{1}{NT}\sum_{t=1}^{T-2}(t-1)^2\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\epsilon}_{t-1} + \frac{2}{NT}\sum_{t=1}^{T-2}(t-1)\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\mu}_{t-1}, \quad (A.23)
\end{aligned}$$

where the first equality follows from the fact that $\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\left(\mathbf{y}_{t-1} - \mathbf{Z}_{t-1}\mathbf{a}_{t-1} - \mathbf{Z}_{t-1}\mathbf{b}_{t-1}\right) = 0$, and the penultimate inequality holds since

$$\begin{aligned}
& \left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right)'\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right) \\
\leq & \lambda_{\max}\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)\left[\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right)'\left(\boldsymbol{\mu}_{t-1} + (t-1)\boldsymbol{\epsilon}_{t-1}\right)\right],
\end{aligned}$$

where $\lambda_{\max}\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)$ denotes the maximum eigenvalue of $\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)$, which is equal to 1 because $\left(\mathbf{I}_N - \mathbf{P}_{t-1}\right)$ is idempotent. And for (A.23), it can be shown that

$$\frac{1}{NT}\sum_{t=1}^{T-2}\boldsymbol{\mu}_{t-1}'\boldsymbol{\mu}_{t-1} = o_p(1),$$

since

$$\begin{aligned}\frac{1}{NT}\sum_{t=1}^{T-2} E\left(\boldsymbol{\mu}_{t-1}'\boldsymbol{\mu}_{t-1}\right) &= \frac{1}{T}\sum_{t=1}^{T-2} E\left(\mu_{it-1}^2\right) \leq \frac{1}{T}\sum_{t=1}^{T-2} O\left(\frac{1}{t}\right) \\ &= O\left(\frac{\log T}{T}\right) \to 0, \quad as\ T\to\infty,\end{aligned}$$

by using the result (A.17). Similarly, we have

$$\frac{1}{NT}\sum_{t=1}^{T-2}(t-1)^2\,\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\epsilon}_{t-1} = o_p(1),$$

since

$$\begin{aligned}\frac{1}{NT}\sum_{t=1}^{T-2}(t-1)^2 E\left(\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\epsilon}_{t-1}\right) &= \frac{1}{T}\sum_{t=1}^{T-2}(t-1)^2 E\left(\epsilon_{it-1}^2\right) \leq \frac{1}{T}\sum_{t=1}^{T-2} O\left(\frac{1}{t}\right) \\ &= O\left(\frac{\log T}{T}\right) \to 0, \quad as\ T\to\infty,\end{aligned}$$

by using the result (A.22). For the cross-product term, it can be shown that by using the Cauchy–Schwarz inequality we have

$$\left|E\left(\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\mu}_{t-1}\right)\right| \leq \sqrt{E\left(\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\epsilon}_{t-1}\right)E\left(\boldsymbol{\mu}_{t-1}'\boldsymbol{\mu}_{t-1}\right)} = O\left(\frac{1}{t^2}\right),$$

which leads to

$$\frac{2}{NT}\sum_{t=1}^{T-2}(t-1)\,\boldsymbol{\epsilon}_{t-1}'\boldsymbol{\mu}_{t-1} = o_p(1),$$

by following the above derivation.

As a result, as $(N,T)\to\infty$, we obtain

$$\begin{aligned}\frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1} &= \frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\mathbf{w}_{t-1} + o_p(1) \\ &\to_p \frac{\sigma_u^2}{1-\gamma^2},\end{aligned}$$

as required. ■

B.1 Asymptotics of GMM based on FOD and FD

Lemma A.3 *Under Assumptions (A1)-(A4) as well as restriction* $\frac{T}{N}\to c\neq 0<\infty$ *as* $(N,T)\to\infty$*, then the following holds for forward demeaning case,* *(a).*

$$\begin{aligned}\frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{y}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{y}_{t-1}^{(f)} &= \frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}+o_p(1)\\ &\rightarrow_p \frac{\sigma_u^2}{1-\gamma^2},\end{aligned}$$

(b).

$$\begin{aligned}\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{y}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{u}_{t}^{(f)} &= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{u}_{t}+o_p(1)\\ &\rightarrow_d N\left(-\frac{\sigma_u^2}{1-\gamma}\sqrt{\frac{T}{N}},\frac{\sigma_u^4}{1-\gamma^2}\right).\end{aligned}$$

Proof. (a) In order to prove this result, using the result (A.4), we have $\mathbf{y}_{t-1}^{(f)}=\mathbf{w}_{t-1}^{(f)}$, where

$$\mathbf{w}_{t-1}^{(f)}=f_{tt}\mathbf{w}_{t-1}+f_{t,t+1}\mathbf{w}_{t}+\cdots+f_{t,T}\mathbf{w}_{T-1}=\sum_{s=t}^{T}f_{ts}\mathbf{w}_{s-1}.$$

Given the above equalities, we can obtain

$$\begin{aligned}\frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{y}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{y}_{t-1}^{(f)} &= \frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}^{(f)}\\ &= \frac{1}{NT}\sum_{t=1}^{T-2}\left(\sum_{s_1=t}^{T}f_{ts_1}\mathbf{w}_{s_1-1}\right)^{\prime}\mathbf{P}_{t-1}\left(\sum_{s_2=t}^{T}f_{ts_2}\mathbf{w}_{s_2-1}\right)\\ &= \frac{1}{NT}\sum_{t=1}^{T-2}\left(f_{tt}\mathbf{w}_{t-1}+\sum_{s_1=t}^{T-1}f_{t,s_1+1}\mathbf{w}_{s_1}\right)^{\prime}\mathbf{P}_{t-1}\left(f_{tt}\mathbf{w}_{t-1}+\sum_{s_2=t}^{T-1}f_{t,s_2+1}\mathbf{w}_{s_2}\right)\\ &= \frac{1}{NT}\sum_{t=1}^{T-2}f_{tt}^2\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}+\frac{2}{NT}\sum_{t=1}^{T-2}f_{tt}\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\sum_{s_2=t}^{T-1}f_{t,s_2+1}\mathbf{w}_{s_2}\\ &\quad+\frac{1}{NT}\sum_{t=1}^{T-2}\sum_{s_1,s_2=t}^{T-1}f_{t,s_1+1}f_{t,s_2+1}\mathbf{w}_{s_1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{s_2}\\ &= A_1+A_2+A_3,\quad\text{say,}\end{aligned}$$

where

$$\begin{aligned}A_1 &= \frac{1}{NT}\sum_{t=1}^{T-2}f_{tt}^2\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}=\frac{1}{NT}\sum_{t=1}^{T-2}\left(1+O\left(\frac{1}{T-t}\right)\right)\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}+o_p(1)\\ &= \frac{1}{NT}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}^{\prime}\mathbf{P}_{t-1}\mathbf{w}_{t-1}+O_p\left(\frac{\log T}{T}\right)\\ &\rightarrow_p \frac{\sigma_u^2}{1-\gamma^2},\end{aligned}$$

as $(N,T)\to\infty$ by using the fact that $f_{tt}=1+O\left(\frac{1}{T-t}\right)$ from (2.3) and the results of Lemma (A.2). For the second term A_2, we have

$$
\begin{aligned}
A_2 &= \frac{2}{NT}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}\left(1+O\left(\frac{1}{T-t}\right)\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\left[-O\left(\frac{1}{T-t}\right)+O\left(\frac{s-t+1}{(T-t)^2}\right)\right]\mathbf{w}_s \\
&= -\frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\sum_{s=t}^{T-1}\mathbf{w}_s+\frac{2}{NT}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{s-t+1}{(T-t)^2}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_s+o_p(1) \\
&= A_{21}+A_{22},
\end{aligned}
$$

where

$$
\begin{aligned}
A_{21} &= -\frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\sum_{s=t}^{T-1}\left(\gamma^{s-t+1}\mathbf{w}_{t-1}+\sum_{l=t}^{s}\gamma^{s-l}\mathbf{u}_l\right) \\
&= -\frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\sum_{s=t}^{T-1}\gamma^{s-t+1}-\frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\sum_{s=t}^{T-1}\sum_{l=t}^{s}\gamma^{s-l}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_l \\
&= -\frac{2\gamma}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\frac{1-\gamma^{T-t}}{1-\gamma}+o_p(1) \\
&= O_p\left(\frac{\log T}{T}\right)=o_p(1),
\end{aligned}
$$

as $T\to\infty$, where the third equality holds simply because $E\left(\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_l\right)=0$ for all $l\geq t$ and by using the results of Lemma (A.1), and the last identity holds by following Lemma (A.2). Similarly,

$$
\begin{aligned}
A_{22} &= \frac{2}{NT}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{s-t+1}{(T-t)^2}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\left(\gamma^{s-t+1}\mathbf{w}_{t-1}+\sum_{l=t}^{s}\gamma^{s-l}\mathbf{u}_l\right) \\
&= \frac{2}{NT}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{s-t+1}{(T-t)^2}\right)\gamma^{s-t+1}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}+o_p(1) \\
&= \frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{(T-t)^2}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\sum_{s=t}^{T-1}(s-t+1)\gamma^{s-t+1}+o_p(1) \\
&= \frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{(T-t)^2}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\left(\frac{\gamma\left(1-\gamma^{T-t}\right)}{(1-\gamma)^2}-\frac{(T-t)\gamma^{T-t+1}}{1-\gamma}\right)+o_p(1) \\
&= -\frac{2}{NT}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\frac{\gamma^{T-t+1}}{1-\gamma}+o_p(1) \\
&= O_p\left(\frac{\log T}{T}\right)=o_p(1),
\end{aligned}
$$

as $T\to\infty$, and then we have $A_2=o_p(1)$.

Finally, for the term A_3, by following the derivation of A_2, we have

$$\begin{aligned}
A_3 &= \frac{1}{NT}\sum_{t=1}^{T-2} O\left(\frac{1}{(T-t)^2}\right) \sum_{s_1,s_2=t}^{T-1} \left(\gamma^{s_1-t+1}\mathbf{w}_{t-1} + \sum_{l_1=t}^{s_1}\gamma^{s_1-l_1}\mathbf{u}_{l_1}\right)' \mathbf{P}_{t-1} \\
&\quad \times \left(\gamma^{s_2-t+1}\mathbf{w}_{t-1} + \sum_{l_2=t}^{s_2}\gamma^{s_2-l_2}\mathbf{u}_{l_2}\right) + o_p(1) \\
&= \frac{1}{NT}\sum_{t=1}^{T-2} O\left(\frac{1}{(T-t)^2}\right) \mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1} \sum_{s_1,s_2=t}^{T-1} \gamma^{s_1-t+1}\gamma^{s_2-t+1} + \\
&\quad + \frac{2}{NT}\sum_{t=1}^{T-2} O\left(\frac{1}{(T-t)^2}\right) \sum_{s_1,s_2=t}^{T-1} \gamma^{s_1-t+1} \sum_{l_2=t}^{s_2}\gamma^{s_2-l_2}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_{l_2} \\
&\quad + \frac{1}{NT}\sum_{t=1}^{T-2} O\left(\frac{1}{(T-t)^2}\right) \sum_{s_1,s_2=t}^{T-1} \sum_{l_1=t}^{s_1}\gamma^{s_1-l_1} \sum_{l_2=t}^{s_2}\gamma^{s_2-l_2}\mathbf{u}_{l_1}'\mathbf{P}_{t-1}\mathbf{u}_{l_2} \\
&= A_{31} + A_{32} + A_{33}, \text{ say,}
\end{aligned}$$

and it is obvious that $A_{31} = o_p(1)$ and $A_{32} = o_p(1)$ by following the above derivation, and for A_{33}, since

$$E\left(\mathbf{u}_{l_1}'\mathbf{P}_{t-1}\mathbf{u}_{l_2}\right) = \begin{cases} 0 & \text{if } l_1 \neq l_2 \\ \sigma_u^2 t & \text{if } l_1 = l_2 \end{cases},$$

under Assumption (A1), then

$$\begin{aligned}
A_{33} &= \frac{1}{NT}\sum_{t=1}^{T-2} O\left(\frac{1}{(T-t)^2}\right) \sum_{s=t}^{T-1}\sum_{l=t}^{s} \gamma^{2s-2l}\mathbf{u}_l'\mathbf{P}_{t-1}\mathbf{u}_l + o_p(1) \\
&= \frac{C}{NT}\sum_{t=1}^{T-2} O\left(\frac{t}{T-t}\right) + o_p(1) \\
&= O_p\left(\frac{\log T}{N}\right) = o_p(1),
\end{aligned}$$

under assumption that $\frac{T}{N} \to c \neq 0$ as $N \to \infty$.

Consequently, combining the above results yields

$$\frac{1}{NT}\sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{y}_{t-1}^{(f)} \to_p \frac{\sigma_u^2}{1-\gamma^2},$$

as required.

(b) To show this result, from the above derivation, we have $\mathbf{u}_t^{(f)} = \sum_{s=t}^{T} f_{ts}\mathbf{u}_s$, then

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{y}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{u}_t^{(f)} \\
=\ &\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}^{(f)\prime}\mathbf{P}_{t-1}\mathbf{u}_t^{(f)} \\
=\ &\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\left(f_{tt}\mathbf{w}_{t-1}+\sum_{s_1=t}^{T-1}f_{t,s_1+1}\mathbf{w}_{s_1}\right)^{\prime}\mathbf{P}_{t-1}\left(f_{tt}\mathbf{u}_t+\sum_{s_2=t+1}^{T}f_{ts_2}\mathbf{u}_{s_2}\right) \\
=\ &\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}f_{tt}^2\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t+\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}f_{tt}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\sum_{s_2=t+1}^{T}f_{ts_2}\mathbf{u}_{s_2} \\
&+\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s_1=t}^{T-1}f_{t,s_1+1}f_{tt}\mathbf{w}_{s_1}'\mathbf{P}_{t-1}\mathbf{u}_t+\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s_1=t}^{T-1}\sum_{s_2=t+1}^{T}f_{t,s_1+1}f_{ts_2}\mathbf{w}_{s_1}'\mathbf{P}_{t-1}\mathbf{u}_{s_2} \\
=\ &B_1+B_2+B_3+B_4,\ \text{say,}
\end{aligned}$$

where the first term B_1 will contribute to the limiting distribution, since $E\left(\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t\right)=0$, and

$$\begin{aligned}
&Var\left(\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t\right)=\frac{1}{NT}\sum_{t=1}^{T-2}E\left(\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t\mathbf{u}_t'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\right) \\
=\ &\frac{\sigma_u^2}{NT}\sum_{t=1}^{T-2}E\left(\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{w}_{t-1}\right)\to\frac{\sigma_u^4}{1-\gamma^2},
\end{aligned}$$

by Lemma (A.2) and $cov(\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t,\mathbf{w}_{s-1}'\mathbf{P}_{s-1}\mathbf{u}_s)=0$, for $t>s$. Consequently, we have

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}f_{tt}^2\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t &= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_t+o_p(1) \\
&\to {}_dN\left(0,\frac{\sigma_u^4}{1-\gamma^2}\right),
\end{aligned}$$

from a standard central limit theorem for autoregressive processes.

For the second term, we have

$$\begin{aligned}
B_2 &= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}f_{tt}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\sum_{s_2=t+1}^{T}f_{ts_2}\mathbf{u}_{s_2} \\
&= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\left(1+O\left(\frac{1}{T-t}\right)\right)\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\sum_{s_2=t+1}^{T}\left[-O\left(\frac{1}{T-t}\right)+O\left(\frac{s-t}{(T-t)^2}\right)\right]\mathbf{u}_{s_2} \\
&= -\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}O\left(\frac{1}{T-t}\right)\sum_{s_2=t+1}^{T}\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_{s_2}+o_p(1) \\
&= o_p(1),
\end{aligned}$$

since $E\left(\mathbf{w}_{t-1}'\mathbf{P}_{t-1}\mathbf{u}_s\right)=0$ for all $s>t$.

For the third term, we have

$$\begin{aligned}
B_3 &= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s_1=t}^{T-1} f_{t,s_1+1}\mathbf{w}_{s_1}'\mathbf{P}_{t-1}f_{tt}\mathbf{u}_t \\
&= \frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}\left[-O\left(\frac{1}{T-t}\right)+O\left(\frac{s+1-t}{(T-t)^2}\right)\right]\left(1+O\left(\frac{1}{T-t}\right)\right)\mathbf{w}_s'\mathbf{P}_{t-1}\mathbf{u}_t \\
&= -\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{1}{T-t}\right)\mathbf{w}_s'\mathbf{P}_{t-1}\mathbf{u}_t-\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{1}{(T-t)^2}\right)\mathbf{w}_s'\mathbf{P}_{t-1}\mathbf{u}_t \\
&\quad +\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\sum_{s=t}^{T-1}O\left(\frac{s+1-t}{(T-t)^2}\right)\mathbf{w}_s'\mathbf{P}_{t-1}\mathbf{u}_t+o_p(1) \\
&= B_{31}+B_{32}+B_{33}+o_p(1),
\end{aligned}$$

note that for $s\geq t$,

$$E\left(\mathbf{w}_s'\mathbf{P}_{t-1}\mathbf{u}_t\right)=\gamma^{s-t}E\left(\mathbf{u}_t'\mathbf{P}_{t-1}\mathbf{u}_t\right)=\gamma^{s-t}E\left(tr\left(\mathbf{u}_t'\mathbf{P}_{t-1}\mathbf{u}_t\right)\right)=\gamma^{s-t}\sigma_u^2 t,$$

then we have

$$\begin{aligned}
B_{31} &= -\frac{\sigma_u^2}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{T-t}\sum_{s=t}^{T-1}\gamma^{s-t}+o_p(1) \\
&= -\frac{\sigma_u^2}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{T-t}\frac{1-\gamma^{T-t}}{1-\gamma}+o_p(1) \\
&= -\frac{\sigma_u^2}{1-\gamma}\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{T-t}+\frac{\sigma_u^2}{1-\gamma}\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{T-t}\gamma^{T-t}+o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
B_{32} &= -\frac{\sigma_u^2}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{(T-t)^2}\sum_{s=t}^{T-1}\gamma^{s-t}+o_p(1) \\
&= -\frac{\sigma_u^2}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{(T-t)^2}\frac{1-\gamma^{T-t}}{1-\gamma}+o_p(1) \\
&= -\frac{\sigma_u^2}{1-\gamma}\frac{1}{\sqrt{NT}}\sum_{t=1}^{T-2}\frac{t}{(T-t)^2}+o_p(1),
\end{aligned}$$

for the last term, we have

$$\begin{aligned}
B_{33} &= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \sum_{s=t}^{T-1} \frac{t(s+1-t)}{(T-t)^2} \gamma^{s-t} + o_p(1) \\
&= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \sum_{s=t}^{T-1} (s+1-t)\, \gamma^{s-t} + o_p(1) \\
&= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \left(\frac{1-\gamma^{T-t}}{(1-\gamma)^2} - \frac{\gamma^{T-t}}{1-\gamma}(T-t) \right) + o_p(1) \\
&= \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t\left(1-\gamma^{T-t}\right)}{(T-t)^2} - \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} \gamma^{T-t} + o_p(1) \\
&= \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} - \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} \gamma^{T-t} + o_p(1).
\end{aligned}$$

For the fourth term, by using the similar argument above, we obtain

$$\begin{aligned}
B_4 &= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \sum_{s_1=t}^{T-1} \sum_{s_2=t+1}^{T} f_{t,s_1+1} f_{ts_2} \mathbf{w}_{s_1}' \mathbf{P}_{t-1} \mathbf{u}_{s_2} \\
&= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \sum_{s_1=t}^{T-1} \sum_{s_2=t+1}^{T} \left(-O\left(\frac{1}{T-t}\right) + O\left(\frac{s_1+1-t}{(T-t)^2}\right) \right) \\
&\quad \times \left(-O\left(\frac{1}{T-t}\right) + O\left(\frac{s_2-t}{(T-t)^2}\right) \right) \mathbf{w}_{s_1}' \mathbf{P}_{t-1} \mathbf{u}_{s_2} \\
&= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \sum_{s_1=t-1}^{T} \sum_{s_2=t+1}^{T} O\left(\frac{1}{(T-t)^2}\right) \mathbf{w}_{s_1}' \mathbf{P}_{t-1} \mathbf{u}_{s_2} + o_p(1) \\
&= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \sum_{s_1 \geq s_2}^{T} O\left(\frac{1}{(T-t)^2}\right) \mathbf{w}_{s_1}' \mathbf{P}_{t-1} \mathbf{u}_{s_2} + o_p(1)
\end{aligned}$$

where the last equality holds since $E\left(\mathbf{w}_{s_1}' \mathbf{P}_{t-1} \mathbf{u}_{s_2}\right) = 0$ for $s_1 < s_2$. And notice that

$$\begin{aligned}
w_{s_1}' P_{t-1} u_{s_2} &= \left(\gamma^{s_1-s_2+1} w_{s_2-1} + \sum_{l=s_2}^{s_1} \gamma^{s_1-l} u_l \right)' P_{t-1} u_{s_2} = \sum_{l=s_2}^{s_1} \gamma^{s_1-l} u_l' P_{t-1} u_{s_2} + o_p(1) \\
&= \gamma^{s_1-s_2} u_{s_2}' P_{t-1} u_{s_2} + o_p(1) \\
&\to {}_p \gamma^{s_1-s_2} \sigma_u^2 t,
\end{aligned}$$

as $s_1 \geq s_2$, then we have

$$
\begin{aligned}
B_4 &= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \sum_{s_1 \geq s_2}^{T} \gamma^{s_1 - s_2} + o_p(1) \\
&= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \sum_{s_2=t+1}^{T} \sum_{s_1=s_2}^{T} \gamma^{s_1 - s_2} + o_p(1) \\
&= \frac{\sigma_u^2}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \sum_{s_2=t+1}^{T} \frac{1-\gamma^{T-s_2}}{1-\gamma} + o_p(1) \\
&= \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t(T-t-1)}{(T-t)^2} - \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \sum_{s_2=t+1}^{T} \gamma^{T-s_2} + o_p(1) \\
&= \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} - \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t(1-\gamma^{T-t})}{(T-t)^2} + o_p(1) \\
&= \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} - \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} + o_p(1)
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
B_3 + B_4 &= -\frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} + \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} \gamma^{T-t} \\
&\quad - \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} + \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} \\
&\quad - \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} \gamma^{T-t} + \frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{T-t} \\
&\quad - \frac{\sigma_u^2}{(1-\gamma)^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} + o_p(1) \\
&= -\frac{\sigma_u^2}{1-\gamma} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \frac{t}{(T-t)^2} + o_p(1) \\
&= -\frac{\sigma_u^2}{1-\gamma} \sqrt{\frac{T}{N}} + o_p(1)
\end{aligned}
$$

As a result, combining these results yields

$$
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T-2} \mathbf{y}_{t-1}^{(f)\prime} \mathbf{P}_{t-1} \mathbf{u}_t^{(f)} \to_d N\left(-\frac{\sigma_u^2}{1-\gamma} \sqrt{\frac{T}{N}}, \frac{\sigma_u^4}{1-\gamma^2} \right),
$$

as required. ∎

Lemma A.4 *Under Assumptions (A1)-(A4) as well as restriction* $\frac{T^3}{N} \to \kappa \neq 0 < \infty$ *as* $(N,T) \to \infty$*, then the following holds for FD case,*

(a).

$$\frac{1}{NT}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{y}_{t-1} \to_p \frac{(1-\gamma)^3}{1+\gamma}\sigma_u^2,$$

(b).

$$\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \to_d N\left(-\frac{(4-\gamma)\,\sigma_u^2}{2}\sqrt{\frac{T^3}{N}}, \frac{2\,(1-\gamma)^4\,(3-\gamma)\sigma_u^4}{1+\gamma}\right).$$

Proof. (a) The proof follows the previous derivation with a little modification to fit the case of FD transformation. We note that

$$\begin{aligned}
& \frac{1}{NT}\sum_{t=3}^{T}\Delta^2\mathbf{y}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{y}_{t-1} \\
= & \frac{1}{NT}\sum_{t=3}^{T}\Delta^2\mathbf{w}_{t-1}'\mathbf{P}_{t-3}\Delta^2\mathbf{w}_{t-1} \\
= & \frac{1}{NT}\sum_{t=3}^{T}\left(\mathbf{w}_{t-1}'-2\mathbf{w}_{t-2}'+\mathbf{w}_{t-3}'\right)\mathbf{P}_{t-3}\left(\mathbf{w}_{t-1}-2\mathbf{w}_{t-2}+\mathbf{w}_{t-3}\right) \\
= & \frac{1}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-1}'\mathbf{P}_{t-3}\mathbf{w}_{t-1}-\frac{4}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-1}'\mathbf{P}_{t-3}\mathbf{w}_{t-2}+\frac{2}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-1}'\mathbf{P}_{t-3}\mathbf{w}_{t-3} \\
& +\frac{4}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-2}'\mathbf{P}_{t-3}\mathbf{w}_{t-2}-\frac{4}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-2}'\mathbf{P}_{t-3}\mathbf{w}_{t-3}+\frac{1}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-3}'\mathbf{P}_{t-3}\mathbf{w}_{t-3} \\
= & \frac{1}{NT}\sum_{t=3}^{T}\left(\gamma^4-4\gamma^3+6\gamma^2-4\gamma+1\right)\mathbf{w}_{t-3}'\mathbf{P}_{t-3}\mathbf{w}_{t-3} \\
& +\frac{1}{NT}\sum_{t=3}^{T}\mathbf{u}_{t-1}'\mathbf{P}_{t-3}\mathbf{u}_{t-1}+\frac{\left(\gamma^2-4\gamma+4\right)}{NT}\sum_{t=3}^{T}\mathbf{u}_{t-2}'\mathbf{P}_{t-3}\mathbf{u}_{t-2} \\
= & \frac{(1-\gamma)^4}{NT}\sum_{t=3}^{T}\mathbf{w}_{t-3}'\mathbf{P}_{t-3}\mathbf{w}_{t-3}+O_p(\frac{T}{N}) \\
\to & _p\frac{(1-\gamma)^3}{1+\gamma}\sigma_u^2,
\end{aligned}$$

under restriction $\frac{T^3}{N} \to \kappa \neq 0 < \infty$ as $N \to \infty$, which holds since, for example,

$$
\begin{aligned}
& \frac{1}{NT}\sum_{t=3}^{T}\mathbf{w}'_{t-1}\mathbf{P}_{t-3}\mathbf{w}_{t-3} \\
= & \frac{1}{NT}\sum_{t=3}^{T}\left(\gamma^2\mathbf{w}_{t-3}+\mathbf{u}_{t-1}+\gamma\mathbf{u}_{t-2}\right)'\mathbf{P}_{t-3}\mathbf{w}_{t-3} \\
= & \frac{1}{NT}\sum_{t=3}^{T}\gamma^2\mathbf{w}'_{t-3}\mathbf{P}_{t-3}\mathbf{w}_{t-3}+\frac{1}{NT}\sum_{t=3}^{T}\mathbf{u}'_{t-1}\mathbf{P}_{t-3}\mathbf{w}_{t-3}+\frac{1}{NT}\sum_{t=3}^{T}\gamma\mathbf{u}'_{t-2}\mathbf{P}_{t-3}\mathbf{w}_{t-3},
\end{aligned}
$$

where the last two terms can be shown to be zero since $E\left(\mathbf{u}'_{t-1}\mathbf{P}_{t-3}\mathbf{w}_{t-3}\right)=0$, and

$$
\begin{aligned}
Var\left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{u}'_{t-1}\mathbf{P}_{t-3}\mathbf{w}_{t-3}\right) &= \frac{1}{N^2T^2}\sum_{t=3}^{T}E\left(\mathbf{w}'_{t-3}\mathbf{P}_{t-3}\mathbf{u}_{t-1}\mathbf{u}'_{t-1}\mathbf{P}_{t-3}\mathbf{w}_{t-3}\right) \\
&= \frac{\sigma_u^2}{N^2T^2}\sum_{t=3}^{T}E\left(\mathbf{w}'_{t-3}\mathbf{P}_{t-3}\mathbf{w}_{t-3}\right) \\
&= o\left(1\right).
\end{aligned}
$$

Similar argument can be applied to other terms.

(b) To show this results, we have

$$
\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\Delta^2\mathbf{y}'_{t-1}\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \\
= & \frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\Delta^2\mathbf{w}'_{t-1}\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \\
= & \frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\left(\mathbf{w}'_{t-1}-2\mathbf{w}'_{t-2}+\mathbf{w}'_{t-3}\right)\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \\
= & \frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\left(\gamma^2\mathbf{w}'_{t-3}+\gamma\mathbf{u}'_{t-2}+\mathbf{u}'_{t-1}-2\gamma\mathbf{w}'_{t-3}-2\mathbf{u}'_{t-2}+\mathbf{w}'_{t-3}\right)\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \\
= & \frac{(1-\gamma)^2}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{w}'_{t-3}\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t+\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\left(\gamma\mathbf{u}'_{t-2}+\mathbf{u}'_{t-1}-2\mathbf{u}'_{t-2}\right)\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \\
= & H_1+H_2,
\end{aligned}
$$

since $\mathbf{w}_{t-1}=\gamma\mathbf{w}_{t-2}+\mathbf{u}_{t-1}$. Then the first term H_1 will contribute to the limiting distribution with

$$
H_1=\frac{(1-\gamma)^2}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{w}'_{t-3}\mathbf{P}_{t-3}\Delta^2\mathbf{u}_t \to_d N\left(0,\frac{2\left(1-\gamma\right)^4\left(3-\gamma\right)\sigma_u^4}{1+\gamma}\right),
$$

since $E(H_1) = 0$ and

$$
\begin{aligned}
Var(H_1) &= \frac{(1-\gamma)^4}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \Delta^2 \mathbf{u}_t \Delta^2 \mathbf{u}_t' \mathbf{P}_{t-3} \mathbf{w}_{t-3}\right) \\
&\quad + \frac{2(1-\gamma)^4}{NT} \sum_{s>t}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3}' \Delta^2 \mathbf{u}_t \Delta^2 \mathbf{u}_s' \mathbf{P}_{s-3} \mathbf{w}_{s-3}\right)
\end{aligned}
$$

where the first term equals to

$$
\begin{aligned}
& \frac{(1-\gamma)^4}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} (\mathbf{u}_t - 2\mathbf{u}_{t-1} + \mathbf{u}_{t-2}) (\mathbf{u}_t - 2\mathbf{u}_{t-1} + \mathbf{u}_{t-2})' \mathbf{P}_{t-3} \mathbf{w}_{t-3}\right) \\
=\ & \frac{(1-\gamma)^4}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \left(\mathbf{u}_t \mathbf{u}_t' + 4\mathbf{u}_{t-1} \mathbf{u}_{t-1}' + \mathbf{u}_{t-2} \mathbf{u}_{t-2}'\right) \mathbf{P}_{t-3} \mathbf{w}_{t-3}\right) \\
=\ & \frac{6\sigma_u^2 (1-\gamma)^4}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \mathbf{w}_{t-3}\right) = \frac{6(1-\gamma)^4 \sigma_u^4}{1-\gamma^2} + o(1),
\end{aligned}
$$

and for the second term, we have

$$
\begin{aligned}
& \frac{2(1-\gamma)^4}{NT} \sum_{t=3}^{T} \sum_{s=t+1}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \Delta^2 \mathbf{u}_t \Delta^2 \mathbf{u}_s' \mathbf{P}_{s-3} \mathbf{w}_{s-3}\right) \\
=\ & \frac{2(1-\gamma)^4}{NT} \sum_{t=3}^{T} [E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \Delta^2 \mathbf{u}_t \Delta^2 \mathbf{u}_{t+1}' \mathbf{P}_{t-2} \mathbf{w}_{t-2}\right) + E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \Delta^2 \mathbf{u}_t \Delta^2 \mathbf{u}_{t+2}' \mathbf{P}_{t-1} \mathbf{w}_{t-1}\right)] \\
=\ & -\frac{8(1-\gamma)^4 \sigma_u^2}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \mathbf{w}_{t-2}\right) + \frac{2(1-\gamma)^4 \sigma_u^2}{NT} \sum_{t=3}^{T} E\left(\mathbf{w}_{t-3}' \mathbf{P}_{t-3} \mathbf{w}_{t-1}\right) \\
=\ & -\frac{8\gamma (1-\gamma)^4 \sigma_u^4}{1-\gamma^2} + \frac{2\gamma^2 (1-\gamma)^4 \sigma_u^4}{1-\gamma^2} + o(1),
\end{aligned}
$$

where the last equality holds by Lemma (A.2). Consequently, we have

$$
\begin{aligned}
Var(H_1) &= \frac{6\sigma_u^4 (1-\gamma)^4}{1-\gamma^2} - \frac{8\gamma\sigma_u^4 (1-\gamma)^4}{1-\gamma^2} + \frac{2\gamma^2 \sigma_u^4 (1-\gamma)^4}{1-\gamma^2} + o(1) \\
&= \frac{2(1-\gamma)^4 (3-\gamma)\sigma_u^4}{1+\gamma} + o(1).
\end{aligned}
$$

And H_2 will contribute to the asymptotic bias with

$$\begin{aligned}
H_2 &= \frac{1}{\sqrt{NT}} \sum_{t=3}^{T} \left(\gamma \mathbf{u}_{t-2}' + \mathbf{u}_{t-1}' - 2\mathbf{u}_{t-2}'\right) \mathbf{P}_{t-3} \left(\mathbf{u}_t - 2\mathbf{u}_{t-1} + \mathbf{u}_{t-2}\right) \\
&= \frac{\gamma - 2}{\sqrt{NT}} \sum_{t=3}^{T} \mathbf{u}_{t-2}' \mathbf{P}_{t-3} \mathbf{u}_{t-2} - \frac{2}{\sqrt{NT}} \sum_{t=3}^{T} \mathbf{u}_{t-1}' \mathbf{P}_{t-3} \mathbf{u}_{t-1} + o_p(1) \\
&= -\frac{(4-\gamma)\,\sigma_u^2}{\sqrt{NT}} \sum_{t=3}^{T} t + o_p(1) \\
&= -\frac{(4-\gamma)\,\sigma_u^2}{2} \sqrt{\frac{T^3}{N}} + o_p(1).
\end{aligned}$$

Combining these results yields

$$\frac{1}{\sqrt{NT}} \sum_{t=3}^{T} \Delta^2 \mathbf{y}_{t-1}' \mathbf{P}_{t-3} \Delta^2 \mathbf{u}_t \rightarrow_d N\left(-\frac{(4-\gamma)\,\sigma_u^2}{2} \sqrt{\frac{T^3}{N}}, \frac{2\,(1-\gamma)^4\,(3-\gamma)\sigma_u^4}{1+\gamma}\right).$$

■

B.2 Asymptotics of Simple IV based on FOD and FD

Lemma A.5 *Under Assumptions (A1)-(A4) and as $(N,T) \to \infty$, then the following holds for FOD case*

(a).

$$\frac{1}{NT} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{y}_{t-1}^{(f)} \rightarrow_p \frac{\sigma_u^2}{1+\gamma},$$

(b).

$$\frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{u}_t^{(f)} \rightarrow_d N\left(0, \sigma_u^2 \left(\frac{\sigma_\delta^2}{(1-\gamma)^2} + \frac{2\sigma_u^2}{1+\gamma}\right)\right).$$

Proof. (a) In order to prove this result, using the result (A.4), we have $\mathbf{y}_{t-1}^{(f)} = \mathbf{w}_{t-1}^{(f)}$, where

$$\mathbf{w}_{t-1}^{(f)} = f_{tt} \mathbf{w}_{t-1} + f_{t,t+1} \mathbf{w}_t + \cdots + f_{t,T} \mathbf{w}_{T-1} = \sum_{s=t}^{T} f_{ts} \mathbf{w}_{s-1}.$$

Also, we have $\Delta \mathbf{y}_t = \boldsymbol{\delta}^* + \Delta \mathbf{w}_t$ from (A.1) where $\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_N^*)'$ and $\Delta \mathbf{w}_t = (\Delta w_{1t}, \ldots, \Delta w_{Nt})'$.

Given the above results, we can obtain

$$
\begin{aligned}
\frac{1}{NT}\sum_{t=2}^{T-2}\Delta\mathbf{y}_{t-1}'\mathbf{y}_{t-1}^{(f)} &= \frac{1}{NT}\sum_{t=2}^{T-2}\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-1}'\right)\left(\sum_{s=t}^{T}f_{ts}\mathbf{w}_{s-1}\right)\\
&= \frac{1}{NT}\sum_{t=2}^{T-2}\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-1}'\right)\left(f_{tt}\mathbf{w}_{t-1}+\sum_{s=t}^{T}f_{t,s+1}\mathbf{w}_{s}\right)\\
&= \frac{1}{NT}\sum_{t=2}^{T-2}\Delta\mathbf{w}_{t-1}'\left(f_{tt}\mathbf{w}_{t-1}+\sum_{s=t}^{T}f_{t,s+1}\mathbf{w}_{s}\right)+o_p\left(1\right),
\end{aligned}
$$

where the last equality holds by using the construction of (A.2) and Assumption (A1). By using f_{ts} $(s \geq t)$ defined in (2.3), it can be shown that

$$
\begin{aligned}
&\frac{1}{NT}\sum_{t=2}^{T-2}\left(f_{tt}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\sum_{s=t}^{T}f_{t,s+1}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{s}\right)\\
=\ &\frac{1}{NT}\sum_{t=2}^{T-2}f_{tt}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T}f_{t,s+1}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{s}\\
=\ &\frac{1}{NT}\sum_{t=2}^{T-2}\left(1+O\left(\frac{1}{T-t}\right)\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T}\left(-O\left(\frac{1}{T-t}\right)+O\left(\frac{s-t+1}{(T-t)^2}\right)\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{s}\\
=\ &\frac{1}{NT}\sum_{t=2}^{T-2}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\frac{1}{NT}\sum_{t=2}^{T-2}O\left(\frac{1}{T-t}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}\\
&-\frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T}O\left(\frac{1}{T-t}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{s}+\frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T}O\left(\frac{s-t+1}{(T-t)^2}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{s}\\
=\ &I_1+I_2+I_3+I_4.
\end{aligned}
$$

For the first term, noticing that w_{it} is an AR(1) process in (A.2), we have

$$
\begin{aligned}
I_1 &= \frac{1}{NT}\sum_{t=2}^{T-2}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}=\frac{1}{NT}\sum_{t=2}^{T-2}\left(\mathbf{w}_{t-1}-\mathbf{w}_{t-2}\right)'\mathbf{w}_{t-1}\\
&= \frac{1}{NT}\sum_{t=2}^{T-2}\left(\mathbf{w}_{t-1}'\mathbf{w}_{t-1}-\mathbf{w}_{t-2}'\mathbf{w}_{t-1}\right)=\frac{\sigma_u^2}{1+\gamma}+o_p(1).
\end{aligned}
$$

By using the similar argument above, we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{NT}\sum_{t=2}^{T-2}O\left(\frac{1}{T-t}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}\\
&= \frac{\sigma_u^2}{1+\gamma}\frac{1}{T}\sum_{t=2}^{T-2}O\left(\frac{1}{T-t}\right)+o_p(1)=O_p\left(\frac{\log T}{T}\right),
\end{aligned}
$$

and

$$\begin{aligned}
I_3 &= -\frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T} O\left(\frac{1}{T-t}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_s \\
&= -\frac{1}{NT}\sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right)\sum_{s=t}^{T}\left(\gamma^{s-t+1}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\sum_{l=t}^{s}\gamma^{s-l}\Delta\mathbf{w}_{t-1}'\mathbf{u}_l\right) \\
&= -\frac{1}{NT}\sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right)\sum_{s=t}^{T}\gamma^{s-t+1}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+o_p(1) \\
&= -\frac{\sigma_u^2}{1+\gamma}\frac{\gamma}{1-\gamma}\frac{1}{T}\sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right)+o_p(1)=O_p\left(\frac{\log T}{T}\right),
\end{aligned}$$

where the third equality holds since $E\left(\mathbf{w}_{t-1}'\mathbf{u}_l\right)=0$ for all $l\geq t$. Finally, we have

$$\begin{aligned}
I_4 &= \frac{1}{NT}\sum_{t=2}^{T-2}\sum_{s=t}^{T} O\left(\frac{s-t+1}{(T-t)^2}\right)\Delta\mathbf{w}_{t-1}'\mathbf{w}_s \\
&= \frac{\sigma_u^2}{1+\gamma}\frac{1}{T}\sum_{t=2}^{T-2} O\left(\frac{1}{(T-t)^2}\right)\sum_{s=t}^{T}(s-t+1)\,\gamma^{s-t+1}+o_p(1) \\
&= \frac{\sigma_u^2}{1+\gamma}\frac{1}{T}\sum_{t=2}^{T-2} O\left(\frac{1}{(T-t)^2}\right)\left(\frac{\gamma\left(1-\gamma^{T-t+1}\right)}{(1-\gamma)^2}-\frac{(T-t+1)\,\gamma^{T-t+2}}{1-\gamma}\right)+o_p(1) \\
&= \frac{\gamma\sigma_u^2}{(1+\gamma)(1-\gamma)^2}\frac{1}{T}\sum_{t=2}^{T-2}\frac{1-\gamma^{T-t+1}}{(T-t)^2}-\frac{\sigma_u^2}{1+\gamma}\frac{1}{1-\gamma}\frac{1}{T}\sum_{t=2}^{T-2}\frac{\gamma^{T-t+2}}{T-t}+o_p(1) \\
&= O_p(\frac{1}{T}),
\end{aligned}$$

where the last equality holds by noticing that $\sum_{t=2}^{T-2}\frac{1-\gamma^{T-t+1}}{(T-t)^2}$ and $\sum_{t=2}^{T-2}\frac{\gamma^{T-t+2}}{T-t}$ are convergent as $|\gamma|<1$. Combining these results, we can obtain

$$\frac{1}{NT}\sum_{t=2}^{T-2}\left(f_{tt}\Delta\mathbf{w}_{t-1}'\mathbf{w}_{t-1}+\sum_{s=t}^{T}f_{t,s+1}\Delta\mathbf{w}_{t-1}'\mathbf{w}_s\right)=\frac{\sigma_u^2}{1+\gamma}+O_p\left(\frac{\log T}{T}\right).$$

As a result, we have

$$\frac{1}{NT}\sum_{t=2}^{T-2}\Delta\mathbf{y}_{t-1}'\mathbf{y}_{t-1}^{(f)}\to_p\frac{\sigma_u^2}{1+\gamma},$$

as $(N,T)\to\infty$.

(b) To this end, we observe that

$$
\begin{aligned}
& \frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}' \mathbf{u}_t^{(f)} \\
= \;& \frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \left(f_{tt} \mathbf{u}_t + \sum_{s=t+1}^{T} f_{ts} \mathbf{u}_s \right) \\
= \;& \frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \mathbf{u}_t - \frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right) \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \sum_{s=t+1}^{T} \mathbf{u}_s + o_p(1),
\end{aligned}
$$

where the last equality holds by using the properties of f_{st} in (2.3). For the second term, it is obvious that the expectation is zero and the variance is given by

$$
\begin{aligned}
& \frac{1}{NT} \sum_{s,t=2}^{T-2} E\left(O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right) \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \sum_{s_1=s+1}^{T} \mathbf{u}_{s_1}' \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{s-1}\right) \right) \\
= \;& \frac{1}{NT} \sum_{t=2}^{T-2} E\left(O\left(\frac{1}{(T-t)^2}\right) \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \mathbf{u}_{t_1}' \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right) \right) \\
& + \frac{2}{NT} \sum_{s<t} E\left(O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right) \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \sum_{s_1=s+1}^{T} \mathbf{u}_{s_1}' \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{s-1}\right) \right),
\end{aligned}
$$

where, for $t = s$, we have

$$
\begin{aligned}
& \frac{1}{NT} \sum_{t=2}^{T-2} O\left(\frac{1}{(T-t)^2}\right) E\left(\left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' E_t\left(\sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \mathbf{u}_{t_1}' \right) \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right) \right) \\
= \;& \frac{1}{NT} \sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right) E\left(\left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right)' \left(\boldsymbol{\delta}^* + \Delta \mathbf{w}_{t-1}\right) \right) \\
= \;& \left(\frac{\sigma_\delta^2}{(1-\gamma)^2} + \frac{2\sigma_u^2}{1+\gamma} \right) \frac{1}{T} \sum_{t=2}^{T-2} O\left(\frac{1}{T-t}\right) \\
= \;& O\left(\frac{\log T}{T}\right),
\end{aligned}
$$

and for $t > s$,

$$
\begin{aligned}
& E\left(O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right)\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} \sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \sum_{s_1=s+1}^{T} \mathbf{u}_{s_1}^{\prime}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{s-1}\right)\right) \\
= & E\left(O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right)\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} E_t\left(\sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \sum_{s_1=s+1}^{T} \mathbf{u}_{s_1}^{\prime}\right)\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{s-1}\right)\right) \\
= & O\left(\frac{1}{T-s}\right) \sigma_u^2 E\left(\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{s-1}\right)\right) \\
= & O\left(\frac{1}{T-s}\right) N \sigma_u^2\left(\frac{\sigma_\delta^2}{(1-\gamma)^2}-\frac{\gamma^{t-s-1}(1-\gamma) \sigma_u^2}{1+\gamma}\right),
\end{aligned}
$$

then

$$
\begin{aligned}
& \frac{1}{NT} \sum_{s,t=2}^{T-2} E\left(O\left(\frac{1}{T-t}\right) O\left(\frac{1}{T-s}\right)\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} \sum_{t_1=t+1}^{T} \mathbf{u}_{t_1} \sum_{s_1=s+1}^{T} \mathbf{u}_{s_1}^{\prime}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{s-1}\right)\right) \\
= & \frac{2\sigma_u^2}{T} \sum_{s<t} O\left(\frac{1}{T-s}\right)\left(\frac{\sigma_\delta^2}{(1-\gamma)^2}-\frac{\gamma^{t-s-1}(1-\gamma) \sigma_u^2}{1+\gamma}\right)+O\left(\frac{\log T}{T}\right) \\
= & O\left(\frac{\log T}{T}\right) \rightarrow_p 0,
\end{aligned}
$$

as $(N,T) \rightarrow \infty$.

Consequently, we have

$$
\frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}^{\prime} \mathbf{u}_t^{(f)}=\frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} \mathbf{u}_t+o_p(1),
$$

and the first term will contribute to the limiting distribution with zero mean and the following asymptotic variance

$$
\begin{aligned}
Var\left(\frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} \mathbf{u}_t\right) & = \frac{1}{NT} \sum_{t=2}^{T-2} E\left[\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime} E_t\left(\mathbf{u}_t \mathbf{u}_t^{\prime}\right)\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)\right] \\
& = \frac{\sigma_u^2}{NT} \sum_{t=2}^{T-2} E\left[\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)^{\prime}\left(\boldsymbol{\delta}^*+\Delta \mathbf{w}_{t-1}\right)\right] \\
& \rightarrow \sigma_u^2\left(\frac{\sigma_\delta^2}{(1-\gamma)^2}+\frac{2\sigma_u^2}{1+\gamma}\right),
\end{aligned}
$$

as $(N,T) \rightarrow \infty$, where $E_t(\cdot)$ is the conditional expectation at time t.

Combining the above derivation yields

$$
\frac{1}{\sqrt{NT}} \sum_{t=2}^{T-2} \Delta \mathbf{y}_{t-1}^{\prime} \mathbf{u}_t^{(f)} \rightarrow_d N\left(0, \sigma_u^2\left(\frac{\sigma_\delta^2}{(1-\gamma)^2}+\frac{2\sigma_u^2}{1+\gamma}\right)\right).
$$

■

Lemma A.6 *Under Assumptions (A1)-(A4) and as $(N,T) \to \infty$, then the following holds for FD case,*

(a).

$$\frac{1}{NT}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1} \to_p \frac{(1-\gamma)^2}{1+\gamma}\sigma_u^2,$$

(b)

$$\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t \to_d N\left(0, \frac{2\left(\gamma^2-5\gamma+10\right)\sigma_u^4}{1+\gamma}\right).$$

Proof. (a) In order to prove this result, using the result (A.3), we have $\Delta^2 y_{it} = \Delta^2 w_{it}$, and $\Delta\mathbf{y}_t = \boldsymbol{\delta}^* + \Delta\mathbf{w}_t$, then we can obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1} \\
= \quad & \frac{1}{NT}\sum_{t=4}^{T}\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-3}'\right)\Delta^2\mathbf{w}_{t-1} = \frac{1}{NT}\sum_{t=4}^{T}\Delta\mathbf{w}_{t-3}'\Delta^2\mathbf{w}_{t-1} + o_p(1) \\
= \quad & \frac{1}{NT}\sum_{t=4}^{T}\left(\mathbf{w}_{t-3}'-\mathbf{w}_{t-4}'\right)\left(\mathbf{w}_{t-1}-2\mathbf{w}_{t-2}+\mathbf{w}_{t-3}\right) + o_p(1) \\
= \quad & \frac{1}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-3}'\mathbf{w}_{t-1} - \frac{2}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-3}'\mathbf{w}_{t-2} + \frac{1}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-3}'\mathbf{w}_{t-3} \\
& -\frac{1}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-4}'\mathbf{w}_{t-1} + \frac{2}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-4}'\mathbf{w}_{t-2} - \frac{1}{NT}\sum_{t=4}^{T}\mathbf{w}_{t-4}'\mathbf{w}_{t-3} + o_p(1) \\
= \quad & \frac{\sigma_u^2}{1-\gamma^2}\left(\gamma^2-2\gamma+1-\gamma^3+2\gamma^2-\gamma\right) + o_p(1) \\
\to \quad & {}_p\frac{(1-\gamma)^2}{1+\gamma}\sigma_u^2,
\end{aligned}$$

as $(N,T) \to \infty$.

(b) For this result, we have

$$E\left(\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\right) = 0,$$

by the moment condition (4.7), also, we have

$$
\begin{aligned}
& Var\left(\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\right) \\
= & \frac{1}{NT}\sum_{s,t=4}^{T}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_s'\Delta\mathbf{y}_{s-3}\right) \\
= & \frac{1}{NT}\sum_{t=4}^{T}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_t'\Delta\mathbf{y}_{t-3}\right)+\frac{2}{NT}\sum_{s>t}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_s'\Delta\mathbf{y}_{s-3}\right),
\end{aligned}
$$

for the first term,

$$
\begin{aligned}
& \frac{1}{NT}\sum_{t=4}^{T}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_t'\Delta\mathbf{y}_{t-3}\right) \\
= & \frac{1}{NT}\sum_{t=4}^{T}E\left(\Delta\mathbf{y}_{t-3}'E_t\left(\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_t'\right)\Delta\mathbf{y}_{t-3}\right) \\
= & \frac{6\sigma_u^2}{NT}\sum_{t=4}^{T}E\left[\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-3}'\right)\left(\boldsymbol{\delta}^{*}+\Delta\mathbf{w}_{t-3}\right)\right] \\
\to & \frac{6\sigma_u^2\sigma_\delta^2}{(1-\gamma)^2}+\frac{12\sigma_u^4}{1+\gamma},
\end{aligned}
$$

and for the second term, we have

$$
\begin{aligned}
& \frac{2}{NT}\sum_{s>t}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_s'\Delta\mathbf{y}_{s-3}\right) \\
= & \frac{2}{NT}\sum_{t=4}^{T}E\left(\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_{t+1}'\Delta\mathbf{y}_{t-2}+\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_{t+2}'\Delta\mathbf{y}_{t-1}\right) \\
= & \frac{2}{NT}\sum_{t=4}^{T}\left[-2E\left(\Delta\mathbf{y}_{t-3}'\left(\mathbf{u}_t\mathbf{u}_t'+\mathbf{u}_{t-1}\mathbf{u}_{t-1}'\right)\Delta\mathbf{y}_{t-2}\right)+E\left(\Delta\mathbf{y}_{t-3}'\mathbf{u}_t\mathbf{u}_t'\Delta\mathbf{y}_{t-1}\right)\right]+o(1) \\
= & \frac{2\sigma_u^2}{NT}\sum_{t=4}^{T}\left[-4E\left(\Delta\mathbf{y}_{t-3}'\Delta\mathbf{y}_{t-2}\right)+E\left(\Delta\mathbf{y}_{t-3}'\Delta\mathbf{y}_{t-1}\right)\right]+o(1) \\
= & \frac{2\sigma_u^2}{NT}\sum_{t=4}^{T}\left[-4E\left(\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-3}'\right)\left(\boldsymbol{\delta}^{*}+\Delta\mathbf{w}_{t-2}\right)\right)+E\left(\left(\boldsymbol{\delta}^{*\prime}+\Delta\mathbf{w}_{t-3}'\right)\left(\boldsymbol{\delta}^{*}+\Delta\mathbf{w}_{t-1}\right)\right)\right]+o(1) \\
= & -\frac{6\sigma_u^2\sigma_\delta^2}{(1-\gamma)^2}+\frac{1-\gamma}{1+\gamma}8\sigma_u^4-\frac{1-\gamma}{1+\gamma}2\gamma\sigma_u^4+o(1).
\end{aligned}
$$

As a result, we have

$$
Var\left(\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\right)\to\frac{2\left(\gamma^2-5\gamma+10\right)\sigma_u^4}{1+\gamma},
$$

and

$$\frac{1}{\sqrt{NT}}\sum_{t=4}^{T}\Delta\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t \rightarrow_d N\left(0,\frac{2\left(\gamma^2-5\gamma+10\right)\sigma_u^4}{1+\gamma}\right),$$

as required. ■

C Invalidity of using level lagged variable as IV for simple IV estimation

Take for the simple IV estimator based on FD as an example, if we multiply both sides of (2.8) by y_{is} $(s < t-2)$ and then take expectation, we obtain

$$E\left(y_{it-3}\Delta^2 y_{it}\right) = \gamma E\left(y_{it-3}\Delta^2 y_{i,t-1}\right).$$

Using the notations in (A.1), we have

$$\begin{aligned} E\left(y_{it-3}\Delta^2 y_{it}\right) &= E\left[\left(\alpha_i^* + \delta_i^*(t-3) + w_{it-3}\right)\Delta^2 w_{it}\right] \\ &= E\left(w_{it-3}\Delta^2 w_{it}\right) = \frac{1-\gamma}{1+\gamma}\gamma\sigma_u^2, \end{aligned}$$

where w_{it} is an $AR(1)$ process. Similarly, we have

$$\begin{aligned} E\left(y_{it-3}\Delta^2 y_{i,t-1}\right) &= E\left[\left(\alpha_i^* + \delta_i^*(t-3) + w_{it-3}\right)\Delta^2 w_{i,t-1}\right] \\ &= E\left(w_{it-3}\Delta^2 w_{i,t-1}\right) = \frac{1-\gamma}{1+\gamma}\sigma_u^2. \end{aligned}$$

As a result, we indeed have

$$\gamma = \left[E\left(y_{it-3}\Delta^2 y_{i,t-1}\right)\right]^{-1} E\left(y_{it-3}\Delta^2 y_{it}\right).$$

Consequently, based on FD transformation, the simple IV estimator using one level lag as instrument can be given by

$$\hat{\gamma}_{IV,level}^{FD} = \left(\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)^{-1}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_t,$$

and

$$\sqrt{NT}\left(\hat{\gamma}_{IV,level}^{FD}-\gamma\right) = \left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t.$$

However, for the limit of $\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t$, we have $E\left(\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\right)=0$ and

$$
\begin{aligned}
& Var\left(\frac{1}{\sqrt{NT}}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\right) \\
= \quad & \frac{1}{NT}\sum_{t=3}^{T}E\left(\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_t'\mathbf{y}_{t-3}\right)+\frac{2}{NT}E\left(\sum_{s>t}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_s'\mathbf{y}_{s-3}\right) \\
= \quad & \frac{6\sigma_u^2}{NT}\sum_{t=3}^{T}E\left[\left(\boldsymbol{\alpha}^*+\boldsymbol{\delta}^*(t-3)+\mathbf{w}_{t-3}\right)'\left(\boldsymbol{\alpha}^*+\boldsymbol{\delta}^*(t-3)+\mathbf{w}_{t-3}\right)\right] \\
& +\frac{2}{NT}E\left(\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_{t+1}'\mathbf{y}_{t-2}+\mathbf{y}_{t-3}'\Delta^2\mathbf{u}_t\Delta^2\mathbf{u}_{t+2}'\mathbf{y}_{t-1}\right) \\
= \quad & O\left(T^2\right).
\end{aligned}
$$

Similarly, for the limit of $\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}$, we have

$$
E\left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)=\frac{1-\gamma}{1+\gamma}\sigma_u^2,
$$

and the variance is given by

$$
\begin{aligned}
& Var\left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right) \\
= \quad & E\left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right)^2-\left[\frac{1-\gamma}{1+\gamma}\sigma_u^2\right]^2 \\
= \quad & \frac{1}{N^2T^2}\sum_{t=3}^{T}E\left(\mathbf{y}_{t-3}'\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\mathbf{y}_{t-3}\right)+\frac{2}{N^2T^2}E\left(\sum_{s<t}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{s-1}'\mathbf{y}_{s-3}\right) \\
& -\left[\frac{1-\gamma}{1+\gamma}\sigma_u^2\right]^2,
\end{aligned}
$$

for the first term, we have

$$
\begin{aligned}
& \frac{1}{N^2T^2}\sum_{t=3}^{T} E\left[\left(\boldsymbol{\alpha}^* + \boldsymbol{\delta}^*(t-3) + \mathbf{w}_{t-3}\right)' \Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\left(\boldsymbol{\alpha}^* + \boldsymbol{\delta}^*(t-3) + \mathbf{w}_{t-3}\right)\right] \\
= & \frac{1}{N^2T^2}\sum_{t=3}^{T} E\left[\boldsymbol{\alpha}^{*\prime}\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\boldsymbol{\alpha}^*\right] + \frac{2}{N^2T^2}\sum_{t=3}^{T} E\left[\boldsymbol{\alpha}^{*\prime}\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\boldsymbol{\delta}^*(t-3)\right] \\
& + \frac{2}{N^2T^2}\sum_{t=3}^{T} E\left[\boldsymbol{\alpha}^{*\prime}\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\mathbf{w}_{t-3}\right] + \frac{1}{N^2T^2}\sum_{t=3}^{T} E\left[\mathbf{w}_{t-3}'\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\mathbf{w}_{t-3}\right] \\
& + \frac{1}{N^2T^2}\sum_{t=3}^{T} (t-3)^2 E\left[\boldsymbol{\delta}^{*\prime}\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\boldsymbol{\delta}^*\right] + \frac{2}{N^2T^2}\sum_{t=3}^{T} E\left[\boldsymbol{\delta}^{*\prime}(t-3)\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\mathbf{w}_{t-3}\right] \\
= & \frac{1}{N^2T^2}\sum_{t=3}^{T} (t-3)^2 \boldsymbol{\delta}^{*\prime} E\left[\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{t-1}'\right]\boldsymbol{\delta}^* + o(1) \\
= & O\left(\frac{T}{N}\right).
\end{aligned}
$$

For the second term, similarly, we can obtain

$$
\begin{aligned}
& \frac{2}{N^2T^2}\sum_{s<t}^{T} E\left[\left(\boldsymbol{\alpha}^* + \boldsymbol{\delta}^*(t-3) + \mathbf{w}_{t-3}\right)' \Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{s-1}'\left(\boldsymbol{\alpha}^* + \boldsymbol{\delta}^*(s-3) + \mathbf{w}_{s-3}\right)\right] \\
= & \frac{2}{N^2T^2}\sum_{s<t}^{T} (t-3)(s-3)\boldsymbol{\delta}^{*\prime} E\left(\Delta^2\mathbf{w}_{t-1}\Delta^2\mathbf{w}_{s-1}'\right)\boldsymbol{\delta}^* + o(1) \\
= & O\left(\frac{T}{N}\right),
\end{aligned}
$$

as a result, as long as $\frac{T}{N} \nrightarrow 0$ as $N \to \infty$, we have

$$
Var\left(\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}\right) \nrightarrow 0,
$$

thus, the limit of $\frac{1}{NT}\sum_{t=3}^{T}\mathbf{y}_{t-3}'\Delta^2\mathbf{y}_{t-1}$ is not a constant and will be a random variable. For simple IV estimator based on FOD transformation, if we use one level lag variables y_{is} $(s < t)$ as instrument, similar results can be obtained.

D Additional Simulation Results

Noting that under assumptions A1-A3, we have $E\left(\Delta y_{is}u_{it}^{(f)}\right) = 0$ for any $s < t$ for model (2.6), and $E\left(\Delta y_{is}\Delta^2 u_{it}\right) = 0$ for any $s < t-2$ for model (2.7). As a result, in addition to use level lags as instruments, we can also use first difference lags as instruments for GMM estimators.

Similarly, it is also obvious that we can use the level lagged dependent variable as instrument for the simple IV estimation. Using the same DGP and specification of parameters in the Monte Carlo simulation, we consider two extra estimators, one is the GMM estimation using all available first differenced lags as instruments, and the other is simple IV estimation using only one level lag as instrument. The simulation results are summarized in Table A1-A3.

There are several interesting findings in Table A1-A3. On the one hand, for the GMM estimation using all available first differenced lags as instruments, we note that they perform quite similarly to the GMM using all level lags in terms of median estimates and iqr, i.e., whether using level lags or first differenced lags will not affect the performance of GMM estimation based on either FOD or FD transformed model. On the other, even if the level lagged variables y_{is} also satisfy the orthogonal condition for (2.8) of simple IV estimation, however, it is shown above that the simple IV estimation using one level lag for both FOD and FD transformed model is invalid. This observation is confirmed from the simulation in Table A1-A3, where we can observe the iqr for simple IV estimation increases with the increase of T for a given N, which is against the results when first differenced lag is used as instrument. The non-decreasing observation of simple IV estimation using level lag is the evident that level lag can't be used as instrument for simple IV estimation for dynamic panel with both individual specific effects and heterogenous time trend.

Table A1: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.2$

	N	200				500				800			
T		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.1734	-0.2353	0.2000	0.1957	0.1877	-0.0512	0.2031	0.2010	0.1924	0.0229	0.2004	0.1987
	bias	-0.0266	-0.4353	-0.0000	-0.0043	-0.0123	-0.2512	-0.0031	0.0010	-0.0076	-0.1771	0.0004	-0.0013
	iqr	0.0334	0.0582	0.2217	0.1390	0.0229	0.0488	0.1518	0.0980	0.0183	0.0393	0.1210	0.0799
	median	0.1816	-0.3517	0.2063	0.1992	0.1917	-0.1554	0.2063	0.1999	0.1950	-0.0611	0.1951	0.2006
50	bias	-0.0184	-0.5517	0.0063	-0.0008	-0.0083	-0.3554	0.0063	-0.0001	-0.0050	-0.2611	-0.0049	0.0006
	iqr	0.0189	0.0283	0.2752	0.1398	0.0118	0.0255	0.1966	0.0951	0.0092	0.0239	0.1482	0.0731
	median	0.1852	-0.4647	0.2062	0.1992	0.1934	-0.2834	0.2130	0.2022	0.1959	-0.1798	0.1972	0.2000
100	bias	-0.0148	-0.6647	0.0062	-0.0008	-0.0066	-0.4834	0.0130	0.0022	-0.0041	-0.3798	-0.0028	0.0000
	iqr	0.0111	0.0143	0.4461	0.1511	0.0073	0.0141	0.2732	0.0962	0.0059	0.0127	0.2194	0.0738

Note: 1. "GMM" refers to the GMM estimation using all first difference lags as instruments, and "IV" refers to simple instrumental variable estimation using only one level lag response as instrument.

2. "FOD" refers to forward demeaning, and "FD" refers to double first difference.

3. iqr refers inter quantile range (75%-25%).

Table A2: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.5$

T	N	200				500				800			
		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.4334	-0.4085	0.4976	0.4888	0.4661	-0.2492	0.5019	0.4991	0.4790	-0.1358	0.4954	0.4985
	bias	-0.0666	-0.9085	-0.0024	-0.0112	-0.0339	-0.7492	0.0019	-0.0009	-0.0210	-0.6358	-0.0046	-0.0015
	iqr	0.0449	0.0656	0.3399	0.4123	0.0319	0.0679	0.2289	0.2816	0.0268	0.0635	0.1886	0.2284
	median	0.4653	-0.4548	0.5140	0.4957	0.4840	-0.3256	0.5039	0.5022	0.4899	-0.2254	0.5014	0.5018
50	bias	-0.0347	-0.9548	0.0140	-0.0043	-0.0160	-0.8256	0.0039	0.0022	-0.0101	-0.7254	0.0014	0.0018
	iqr	0.0204	0.0286	0.4069	0.4275	0.0149	0.0331	0.2852	0.2835	0.0113	0.0321	0.2178	0.2298
	median	0.4777	-0.4984	0.5313	0.5038	0.4896	-0.4057	0.5225	0.5068	0.4934	-0.3289	0.4935	0.4942
100	bias	-0.0223	-0.9984	0.0313	0.0038	-0.0104	-0.9057	0.0225	0.0068	-0.0066	-0.8289	-0.0065	-0.0058
	iqr	0.0110	0.0141	0.5013	0.4375	0.0074	0.0144	0.3721	0.3009	0.0059	0.0151	0.2998	0.2370

See notes of Table A1.

Table A3: Median, median-bias and iqr of different estimators of γ when $\gamma = 0.7$

T	N	200				500				800			
		GMM		IV		GMM		IV		GMM		IV	
		FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD	FOD	FD
25	median	0.5372	-0.4946	0.6981	0.5095	0.5961	-0.4570	0.7042	0.6696	0.6234	-0.4259	0.6978	0.6824
	bias	-0.1628	-1.1946	-0.0019	-0.1905	-0.1039	-1.1570	0.0042	-0.0304	-0.0766	-1.1259	-0.0022	-0.0176
	iqr	0.0665	0.0610	0.5459	1.1306	0.0538	0.0621	0.3757	0.8538	0.0461	0.0638	0.2861	0.6831
	median	0.6352	-0.5004	0.7499	0.5220	0.6653	-0.4666	0.7159	0.6457	0.6764	-0.4357	0.7111	0.7043
50	bias	-0.0648	-1.2004	0.0499	-0.1780	-0.0347	-1.1666	0.0159	-0.0543	-0.0236	-1.1357	0.0111	0.0043
	iqr	0.0248	0.0275	0.5588	1.2229	0.0179	0.0290	0.4012	0.9190	0.0152	0.0301	0.3070	0.7585
	median	0.6671	-0.5081	0.7905	0.5318	0.6833	-0.4847	0.7597	0.6797	0.6888	-0.4616	0.7149	0.6667
100	bias	-0.0329	-1.2081	0.0905	-0.1682	-0.0167	-1.1847	0.0597	-0.0203	-0.0112	-1.1616	0.0149	-0.0333
	iqr	0.0112	0.0141	0.5614	1.2426	0.0077	0.0137	0.4563	0.9725	0.0063	0.0142	0.3670	0.7666

See notes of Table A1.